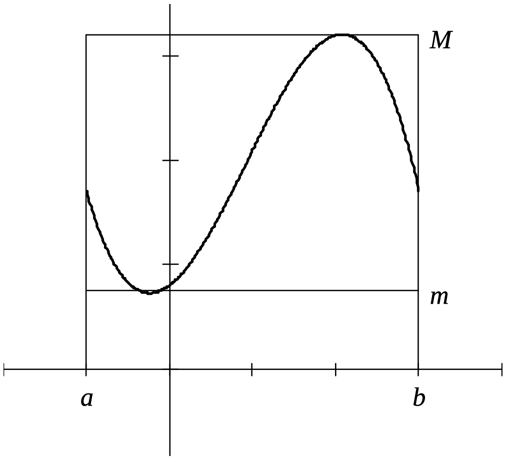
Average Value of a Function

Introduction

Finding the average value of a function is yet another application of the definite integral. You've found averages many times. You add up the individual values and then divide by the number of values and you have the average. Think about doing this with a function defined on an interval. Using the same technique to find the average value of a function leads to problems almost immediately. If you have a function that is continuous on a particular interval, there are an infinite number of values to "add up". Well, even if you could add them up, you would then have to divide by the total number of values to get an average…but there are an infinite number of these values…divide by infinity? No. Instead, the definite integral will be used—after all, we've used it over and over again to add up infinite numbers of objects from rectangles to slices to shells.

An important intermediate result

Consider the following diagram. It shows a function *f* on some closed interval $[a,b]$. The absolute maximum function value on $[a,b]$ is labeled *M* and the absolute minimum function value on $[a,b]$ is labeled *m*.



Note that $M(b-a)$ is the area of the large rectangle bounded by $x=a, x=b,$ the x-axis and $y=M$. Similarly, $m(b-a)$ is the area of the rectangle bounded by $x=a, x=b,$ the x-axis and $y=m$.

Now, $\int_a^b f(x)\,dx$ is the area bounded by $x=a, x=b,$ the x-axis and the function *f*. Clearly,

$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$. At first glance this may seem obvious and unimportant but it is actually a very powerful and important result. Keep in mind that integration is a more complex operation than differentiation. Also keep in mind that finding an antiderivative of a function is not always an easy task. This relationship, $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$, allows us to put an upper and a lower bound on the value of a definite integral without ever integrating! Why would this be important? We've stated several reasons in previous sections, like the difficulty of antidifferentiating some functions. Many (if not most) engineering and scientific applications require only that a particular value fall between certain parameters. For instance, a bridge must be able to withstand a certain minimum load but it is not necessary (and also expensive) to build it to withstand more that a specified maximum load. The bridge engineers are only concerned that the bridge be able to withstand loads between two values—a minimum and a maximum. When parts are built for machinery the same "tolerance" principle is used. A piston, which moves at a high speed inside a cylinder, must be made so that its diameter is small enough to fit inside the cylinder but not so large that it doesn't. As long as the diameter is between two values, a minimum and a maximum, the piston will work.

Using the relationship we just discovered, we can put an upper and lower bound on a definite integral by finding the absolute maximum and absolute minimum values of the function on the interval.

Example 1

Consider the function $f(x) = x^3 - 2x^2 - 1$ on $[0,3]$. Find an interval, which contains the value of $\int_0^3 (x^3 - 2x^2 - 1)\,dx$.

First, we need to find the absolute maximum and absolute minimum function values on $[0,3]$. Taking the derivative yields

$$f'(x) = 3x^2 - 4x$$

Since $f' \,\exists \forall\, x \in [0,3]$ and $f'(x) = 0$ when $x = 0$ or $x = \frac{4}{3}$, the critical numbers occur at $x = 0$ and $x = \frac{4}{3}$. Using the Extreme Value Theorem, we will find function values at the endpoints of the interval and at these critical numbers—the largest value being the absolute maximum function value and the smallest the absolute minimum function value.

$$f(0) = -1$$

$$f(3) = 8$$

$$f\left(\frac{4}{3}\right) = -\frac{59}{27}$$

The absolute maximum is 8 and the absolute minimum is $-\frac{59}{27}$. Multiplying these by $(3-0)$ gives us the interval $\left[-\frac{59}{9}, 24\right]$ or $[-6.556, 24]$.

Thus, without ever integrating, we know that the value of $\int_0^3 (x^3 - 2x^2 - 1)\,dx$ is somewhere in the interval $\left[-\frac{59}{9}, 24\right]$ or $[-6.556, 24]$.

Let's return now to average value. What we actually did was find a function value (*M*), whose product with $(b-a)$ yielded a rectangle whose area was larger than the exact area (given by the definite integral). We also found a function value (*m*), whose product with $(b-a)$ yielded a rectangle whose area was smaller than the value of the definite integral.

Thus, by the Intermediate Value Theorem, there must exist a function value on the interval $[a,b]$, whose product with $(b-a)$ will yield a rectangle whose area is exactly the same as the value of the definite integral! This function value is called the <u>average value</u> of f on $[a,b]$. The theorem, which states this, is called the Mean Value Theorem for Integrals.

The Mean Value Theorem for Integrals

If f is continuous on $[a,b]$, then there exists a z in $[a,b]$ such that

$$f(z)\cdot(b-a)=\int_a^b f(x)\,dx.$$

If we solve this equation for $f(z)$ we obtain

$$f(z)=\frac{\int_a^b f(x)\,dx}{(b-a)}$$

This is the formula we use to calculate the average value of a function on an interval. Sometimes, instead of writing $f(z)$, we will write f_{avg}.

You need to take a moment to think about the result of such a calculation. Since a continuous function has an infinite number of function values on any interval, you are actually finding the average of an infinite number of values! It is literally the average value—in the same way that 6 is the average of 7 and 8!

Example 2

Find the average value of $f(x)=4-x^2$ on $[0,2]$.

$$\begin{aligned} f_{avg} &= \frac{\int_0^2 (4-x^2)\,dx}{2-0} \\ &= \tfrac{1}{2}\left[4x-\tfrac{1}{3}x^3\right]_0^2 \\ &= \tfrac{8}{3} \end{aligned}$$

Example 3

The temperature (Fahrenheit) in a certain city t hours after 9 a.m. was approximated by $T = 50 + 14\sin\frac{p\,t}{12}$. Find the average temperature during the period from 9 a.m. to 9 p.m.

$$\begin{aligned} T_{avg} &= \frac{\int_0^{12}\left[50 + 14\sin\frac{p\,t}{12}\right] dt}{12 - 0} \\ &= \frac{1}{12}\left[50t - \frac{168}{p}\cos\frac{p\,t}{12}\right]_0^{12} \\ &\approx 58.913 \end{aligned}$$

Therefore, the average temperature was 58.913 degrees Fahrenheit.

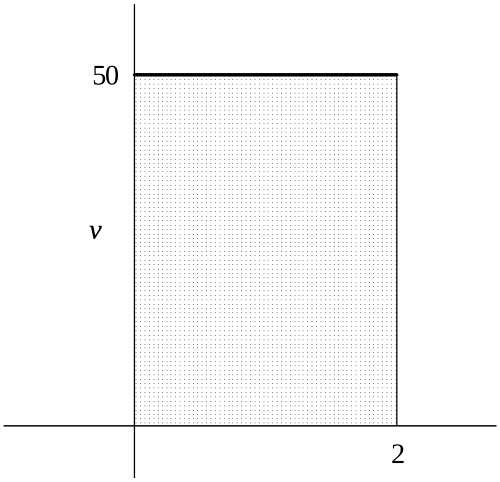
The Definite Integral as an Accumulator

Introduction

What is it that the definite integral really does? We know that it is the limit of a sum of objects as the number of objects goes to infinity. In this sense, the definite integral is an accumulator. When we found areas under curves, the definite integral accumulated an infinite number of areas of rectangles. In volume problems, the definite integral accumulated an infinite number of volumes…either disks, washers, shells or slices. In this section we will discuss other applications of this tremendously important idea…the definite integral as an accumulator.

Distance

Consider a car moving at a constant velocity of 50 miles per hour. If it moves at this velocity for two hours, it will travel 100 miles. Let's look at this problem in terms of area. If the velocity is constant, the velocity function can be written $v(t)=50$. Below is the graph of this function on the interval $[0,2]$. The area under v between 0 and 2 is shaded.

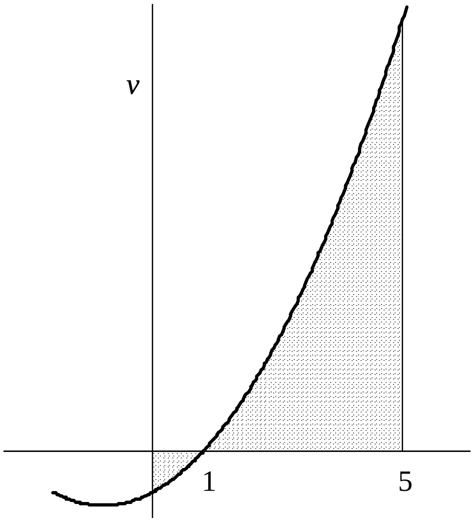


Notice that the area under the curve (the area of the shaded rectangle) is 100. This area could also be expressed as the integral

$$\int_0^2 50\ dx = 50x\Big|_0^2 = 100 - 0 = 100$$

If you integrate a velocity function over a specified interval, it will yield information about the distance the object traveled. We do however, need to be careful. There are two types of "distance traveled"—total distance and net distance. If you measure the distance you travel in one day from the time you get out of bed in the morning until the time you return to your bed at night, your net distance traveled is zero. Your total distance traveled may be one mile, one hundred miles or thousands of miles, but your net distance is zero.

Consider the velocity function $v(t)=3t^2+6t-9$ graphed below. The shaded region is the area bounded by $v(t)$, the $t-$axis, $t=0$ and $t=5$.



The shaded area is a measure of the distance the object travels from $t=0$ to $t=5$. Evaluating the integral $\int_0^5 (3t^2+6t-9)\ dt$ yields a number which is the area above the axis minus the area below the axis. Since the velocity function is below the axis on $(0,1)$, the velocity is negative—the particle is moving "backwards". The small shaded region below the axis is a measure of how far the object traveled in a negative direction. The area above the axis will be the distance the object traveled in the positive direction. In this case, $\int_0^5 (3t^2+6t-9)\ dt$ would yield the net distance the object traveled—not the total distance it traveled.

Given $v(t)=3t^2+6t-9$, how could we calculate total distance? All we need to do is make sure that the negative region is added to the positive region instead of subtracted from it. Remember that taking the absolute value of a function has the effect of reflecting any portion of its graph from below to above the horizontal axis. The total distance traveled can be obtained by evaluating $\int_0^5 \left|3t^2+6t-9\right|\ dt$. To evaluate this integral we need to find the zeros of the function, make a chart to determine where the function is positive or negative, and set up the appropriate integrals. For this function the zeros are $t=-3$ and $t=1$. The graph of the function is below the *t*-axis on $(0,1)$ and above on $(1,5)$. We will add the negative area by taking the opposite of the integral from $t=0$ to $t=1$. To calculate total distance traveled we would evaluate:

$$\int_0^5 \left|3t^2+6t-9\right|\ dt=-\int_0^1 (3t^2+6t-9)\ dt+\int_1^5 (3t^2+6t-9)\ dt$$

If the area under a velocity curve is above the horizontal axis on the entire interval, net distance and total distance will be the same. If the area under the velocity curve is partly above and partly below, we must integrate the absolute value of the velocity function to get total distance.
In summary,

Net distance: $\int_{t_1}^{t_2} v(t)\ dt$

Total distance: $\int_{t_1}^{t_2} |v(t)|\ dt$

If $v(t) > 0\ \forall t \in (t_1, t_2)$,
net distance = total distance.

General applications of the definite integral as an accumulator

Given a function that describes the rate of change in a quantity, its integral over a specified interval will yield the "amount of change". For example:

If $R(t)$ describes the rate of growth of a child in inches per year from birth to age 18, $\int_0^3 R(t)\ dt$ would yield the total number of inches a child grew in the first three years.
If $E(x)$ describes the rate of electricity consumption in a twenty-four hour period in kilowatts per day, $\int_0^7 E(x)\ dx$ would yield the total number of kilowatts consumed in the first seven hours.

Notice that in each example above, we did not use absolute value. For the most part, the only time we need to be concerned about the difference between net and total change in a quantity is when we are dealing with distance problems. It would make little sense for a function that describes a rate of growth in the height of a child, electricity consumption, oil production, etc. to "go negative". The rate of change in these quantities may drop to zero but a negative rate of growth in such quantities would not make any sense.

In general, if we integrate a rate of change, we will get the total amount of change—except for distance problems which may involve either net or total change.