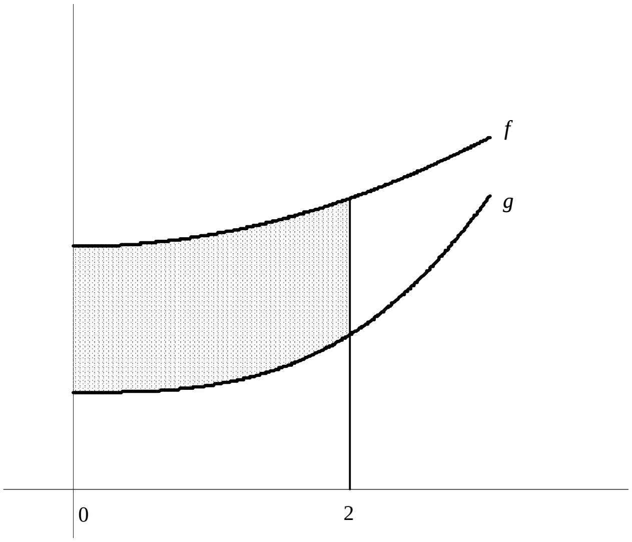
Areas Between Curves

Introduction

We now move on to a wider variety of area problems. Up to this point we have worked with the simplest of area problems—the area under one curve when the area in question lies entirely above the *x*-axis. In order to move forward we need to combine our knowledge of the Fundamental Theorems of Calculus with the properties of the definite integral. Consider the following graph of two functions, *f* and *g*, on the interval [0,2].



Suppose we wanted to find the area between these two curves on the interval [0,2]. The definite integral $\int_0^2 f(x)\,dx$ would yield the area under *f* and the definite integral $\int_0^2 g(x)\,dx$ would yield the area under *g*. Clearly, subtracting the area under *g* from the area under *f* would give us the area between the two curves. This definite integral, $\int_a^b [f(x)-g(x)]\,dx$, will be our basic tool to find area between two curves.

Although the problems will vary widely, we can list several basic problem types:

- finding area under one curve in which the area lies entirely above the *x*-axis
- finding area under one curve in which the area lies entirely or partially below the *x*-axis
- finding area between two curves with the bounds given
- finding area between two curves with the bounds not given

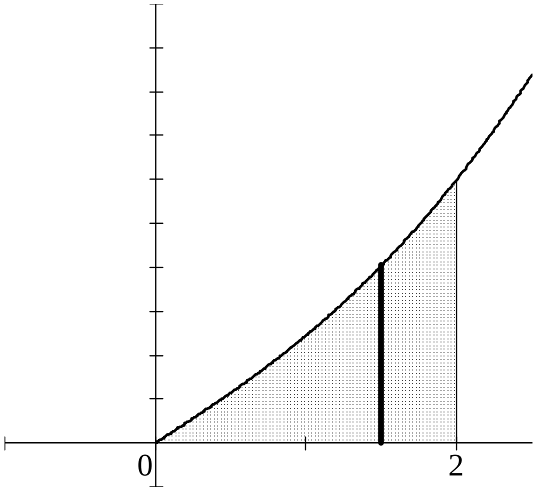
When we speak of the area "under" a curve, we actually mean the area between the function and the *x*-axis or $f(x)=0$. You will not see the term "under" much anymore. Instead, the area we are asked to find will be specified in term of bounds. A problem which we previously stated as "find the area under $f(x)=x^2$ from $x=2$ to $x=5$" will now be stated "Find the area of the region bounded by $f(x)=x^2$, the *x*-axis, $x=2$ and $x=5$". If we consider all area problems as the area between two curves, we can simplify our thinking by consistently saying to ourselves that the area between two curves will be the integration of the "top curve minus the bottom curve". If the problem asks for the area "under" a curve, we think of $f(x)=0$ as the second curve—sometimes it will be the "top" curve and other times it will be the "bottom" curve.

As a general guideline, it is always a good idea to work from a sketch and include it as part of our solution. This sketch should always include the *i*th rectangle drawn inside the requested area. The inclusion of the *i*th rectangle will become even more important as we move on to more complicated problems so make it a habit now.

The Simplest Case—One curve with the area entirely above the *x*-axis

Find the area of the region in the first quadrant bounded by $y = x\sqrt{x^2+5}$, the *x*-axis and $x=2$.

Begin with a sketch of the problem. The dark line represents the *i*th rectangle.



The curve clearly passes through $x=0$, so $x=0$ is the lower bound of the definite integral. The "top" curve is $y = x\sqrt{x^2+5}$ and the "bottom" curve is $y=0$ so the integral we need is

$$A = \int_0^2 \left[\left(x\sqrt{x^2+5} \right) - 0 \right] dx$$

or simply,

$$A = \int_0^2 x\sqrt{x^2+5}\ dx$$

This integral will require us to use substitution.

$$\begin{aligned} u &= x^2+5 \\ du &= 2x\ dx \\ \frac{1}{2}du &= x\ dx \end{aligned} \qquad \begin{aligned} x=0 &\rightarrow u=5 \\ x=2 &\rightarrow u=9 \end{aligned}$$

$$A=\int_0^2 x\sqrt{x^2+5}\ dx=\frac{1}{2}\int_5^9 u^{1/2}\ du$$
$$=\frac{1}{2}\frac{2}{3}u^{3/2}\Big|_5^9$$
$$=\frac{1}{3}9^{3/2}-\frac{1}{3}5^{3/2}$$
$$=\frac{27-5\sqrt{5}}{3}$$

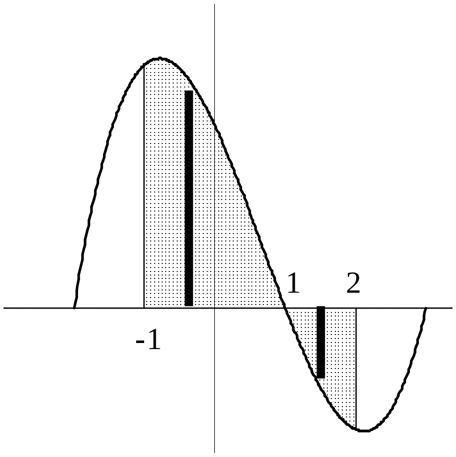
Therefore the area is $\frac{27-5\sqrt{5}}{3}$.

Remember that when you are evaluating definite integrals using substitution, once you have the problem in terms of u, you do not have to go back to putting it in terms of x. Just keep it in u and evaluate.

Area below the *x*-axis

Find the area of the region bounded by $y=x^3-2x^2-5x+6$, the x-axis, $x=-1$ and $x=2$.

Below is a sketch of the problem.



This problem brings up another important aspect of finding areas bounded by the x-axis. It is essential to know the zeros of the function! This will help you determine which areas are above and which are below the axis. The zeros are determined by solving

$$x^3-2x^2-5x+6=0$$

which yields

$$x=-2,\ x=1 \text{ or } x=3$$

On the interval $[-1,2]$, the curve crosses the x-axis at $x=1$. On $[-1,1]$ the curve is above the x-axis and on $[1,2]$ the curve is below the x-axis. In order to get the total area, we will need to subtract the area under the x-axis. The expression which will yield the area is

$$A=\int_{-1}^{1}\left(x^3-2x^2-5x+6\right)dx-\int_{1}^{2}\left(x^3-2x^2-5x+6\right)dx\ .$$

This integral will not require substitution—just a lot of algebra and arithmetic!

$$\begin{aligned} A &= \int_{-1}^{1}\left(x^3-2x^2-5x+6\right)dx-\int_{1}^{2}\left(x^3-2x^2-5x+6\right)dx \\ &= \left[\frac{1}{4}x^4-\frac{2}{3}x^3-\frac{5}{2}x^2+6x\right]_{-1}^{1}-\left[\frac{1}{4}x^4-\frac{2}{3}x^3-\frac{5}{2}x^2+6x\right]_{1}^{2} \\ &= \left[\frac{37}{12}-\frac{-91}{12}\right]-\left[\frac{2}{3}-\frac{37}{12}\right] \\ &= \frac{157}{12} \end{aligned}$$

Therefore the area is $\frac{157}{12}$ square units.

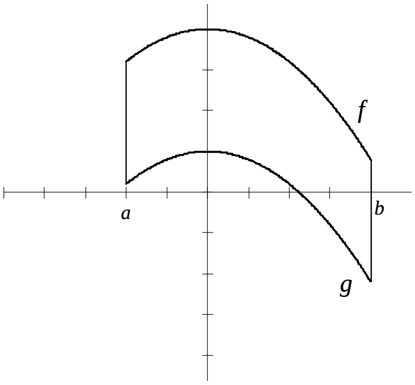
Note: We used a sketch of the curve and the zeros to determine where the curve was above and below the *x*-axis. We could have done this analytically—without a sketch. We know that the only zero on the interval $[-1,2]$ is $x=1$, we could have chosen any *x*-value in the interval $[-1,1]$ like $x=0$ and found the value of the function at $x=0$. This would have resulted in a *y*-coordinate of 6, which is above the *x*-axis, so we would know the curve is above the x-axis on $[-1,1]$. Similarly, we could have found the value of the function at any *x*-value in $[1,2]$, say $x=1.5$ and determined from the *y*-coordinate that the curve was below the x-axis on $[1,2]$.

Area between two curves

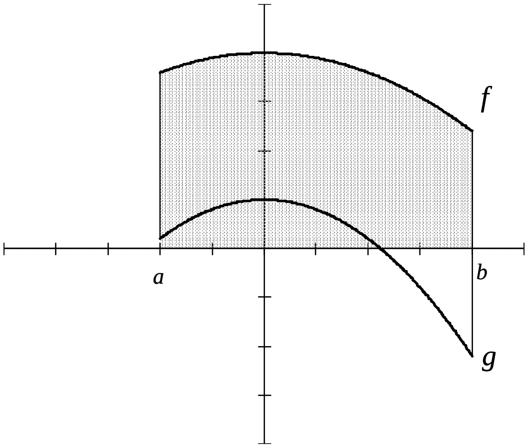
We already know that the area between two curves *f* and *g* from *a* to *b* is given by

$$A=\int_a^b [f(x)-g(x)]\,dx$$

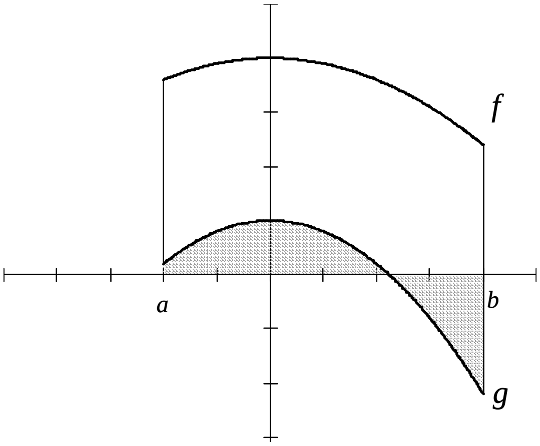
where $f(x)\ge g(x)\,\forall x\in(a,b)$. This integral will always yield the area between two curves—even if all or part of the region is below the x-axis. Consider the diagram below.



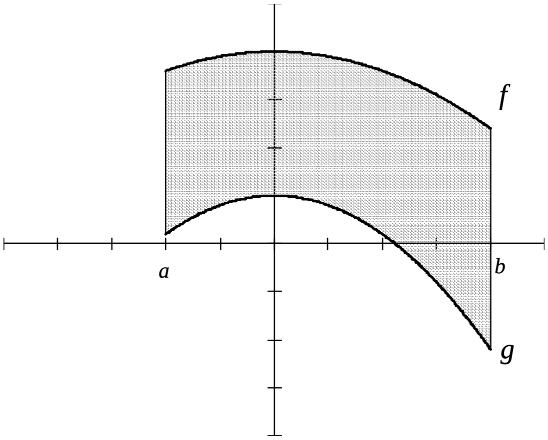
Now, $\int_a^b f(x)\,dx$ will give us the area under f and above the *x*-axis from *a* to *b*. This is the shaded region in the diagram below.



The integral $\int_a^b g(x)\,dx$ will yield the area between the *x*-axis and *g*. This region is shaded in the diagram below.

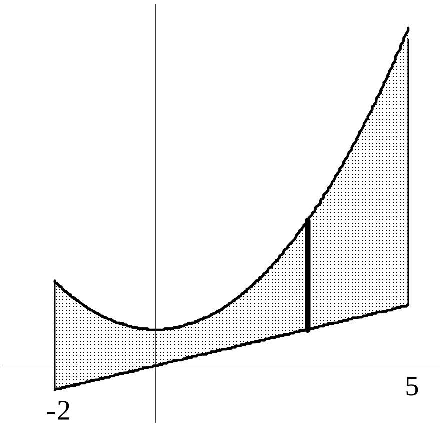


When we subtract this second area from the first an amazing thing happens. We needed to take away the area under g which is above the x-axis. Since this is a positive area, our subtraction does just what we wanted. We also needed to add on the area which is below the x-axis and above g. This is a negative area and when we subtract the two integrals this region gets added back—just want we wanted! Subtracting the negative area added it back on! The area resulting from $A = \int_a^b [f(x) - g(x)]\,dx$ is shown below.



Area between curves—bounds given

Find the area of the region bounded by $y = x^2 + 3$, $y = x$, $x = -2$ and $x = 5$. Below is a sketch of the problem.



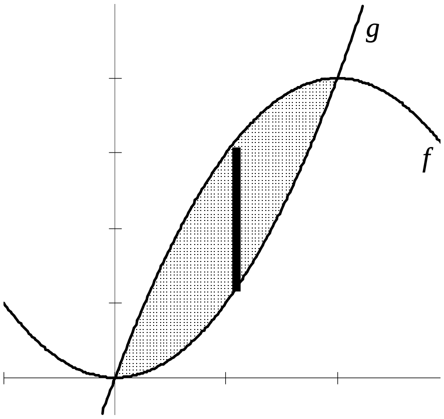
This problem is relatively straightforward to set up. Using our "top curve minus bottom curve" idea we obtain the integral

$$A = \int_{-2}^{5} [(x^2 + 3) - (x)]\, dx\,.$$

Evaluating this integral yields an area of $\frac{329}{6}$ square units.

Area between two curves—bounds not given

Let's find the area of the region bounded by $f(x) = -x^2 + 4x$ and $g(x) = x^2$. Notice that no other bounds are given—we'll have to find them. Let's start with a sketch.



Our first step is to find the bounds. We do this by finding the intersections of the two curves.

$$-x^2 + 4x = x^2$$
$$2x^2 - 4x = 0$$
$$2x(x-2) = 0$$
$$x = 0 \text{ or } x = 2$$

From our sketch we see that on the interval $[0,2]$ f is the "top" curve and g is the "bottom" curve. Our integral to find the area becomes

$$A = \int_0^2 [(-x^2 + 4) - (x^2)]\, dx .$$

Evaluating this integral we get an area of $\frac{8}{3}$ square units.

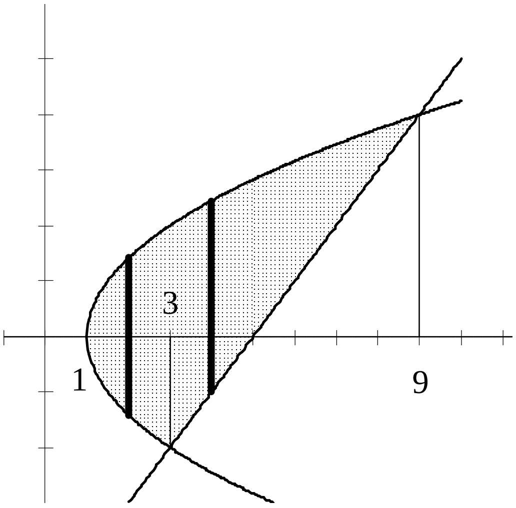
Horizontal and vertical elements

All of the problem we've done so far have been done in terms of *x*—we were always able to set up a relatively simple integral in terms of *x*. Our *i*th rectangle (our element) was always placed vertically in the area. It's width was *dx* and this told us that the problem would be set up in terms of *x*. Some problems will be easier if we set them up horizontally—in terms of *y* instead of *x*. We'll do the next problem both ways…first in terms of *x* (vertically) and then in terms of *y* (horizontally).

Let's find the area of the region bounded by $y^2 = 2x - 2$ and $y = x - 5$. On the sketch below we see that when we try to get the problem in terms of x, the curve $y^2 = 2x - 2$ actually becomes two curves: $y = \sqrt{2x-2}$ (the top half) and $y = -\sqrt{2x-2}$ (the lower half). The intersections of the two curves were found by solving each equation for *x* and setting them equal.

$$y + 5 = \frac{y^2 + 2}{2}$$
$$y = -2 \text{ or } y = 4$$

This gives us the intersections $(3,-2)$ and $(9,4)$. We also determined that $y^2 = 2x - 2$ crosses the *x*-axis at $x = 1$.

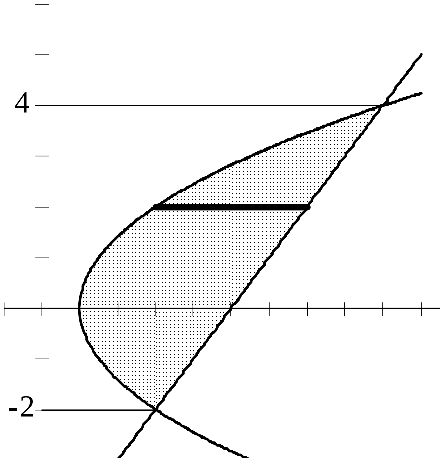


Also notice on the sketch that as the element "moves" from left to right through the area, the bottom of the element "switches" curves. From $x=1$ to $x=3$ the bottom of the element is on $y=-\sqrt{2x-2}$. From $x=3$ to $x=9$ the bottom of the element is on $y=x-5$. The means that the "bottom" curves changes. Calculating the area will require the following two integrals:

$$A=\int_1^3\left[\left(\sqrt{2x-2}\right)-\left(-\sqrt{2x-2}\right)\right]dx+\int_3^9\left[\left(\sqrt{2x-2}\right)-(x-5)\right]dx$$

This yeilds an area of 18 square units. (It also required substitution.)

Let's attempt the problem again—looking at the problem horizontally. Below is our new sketch.



Notice that as the element moves through the area from $y=-2$ to $y=4$, it never "switches" curves. The "top" is always on $x=y+5$ and the "bottom" is always on the parabola $x=\dfrac{y^2+2}{2}$. This makes the integral we must use to find the area much simpler. The integral is now

$$A=\int_{-2}^{4}\left[(y+5)-\left(\frac{y^2+2}{2}\right)\right]dy.$$

Rewriting this integral as $A=\int_{-2}^{4}\left[(y+5)-\left(\tfrac{1}{2}y^2+1\right)\right]dy$ will make it even easier to evaluate. Again, we obtain an area of 18 square units.

Summary

When finding the area between two curves keep in mind the following:

- Draw a sketch.
- Draw the *i*th rectangle on your sketch.
- If you are not given the bounds you must find them—usually by intersecting the two curves.
- Find all zeros of the functions in the problem—especially if the zeros are inside the area in question.
- If you set up a problem vertically (usually our first choice) and you see that the element switches curves, try doing the problem horizontally.

Volumes of Solids with Known Cross Sections

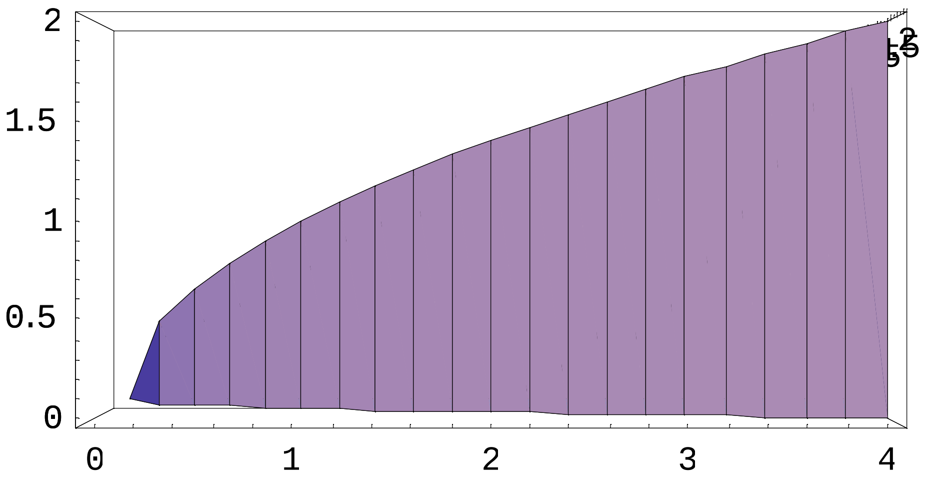
Introduction

Our basic process for finding areas under (or between) curves is to find an expression for the area of an arbitrary rectangle, let the number of rectangles go to infinity and then sum up this infinite number of rectangles. We will now extend this idea into three dimensions to find the volumes of various solids. Our volume problems can be categorized into two groups: volumes of objects with known cross sections and volumes generated when a region is rotated about some vertical or horizontal line. This lesson will address the former.

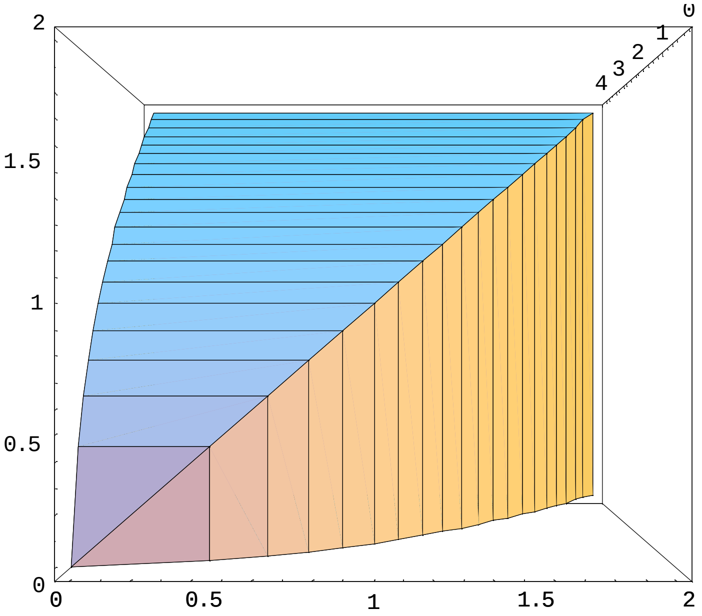
Volumes by "Slicing"

The process of finding the volume of a solid with a known cross section is, at least theoretically, fairly straightforward. We will find an expression for the volume of a slice of the solid and then integrate this expression over a specified interval. Remember, the definite integral essentially allows us to sum an infinite number of objects. It's sort of like finding the volume of a loaf of bread by finding the volume of one very thin slice and then adding up an infinite number of these slices—all packed between the two ends of the loaf.

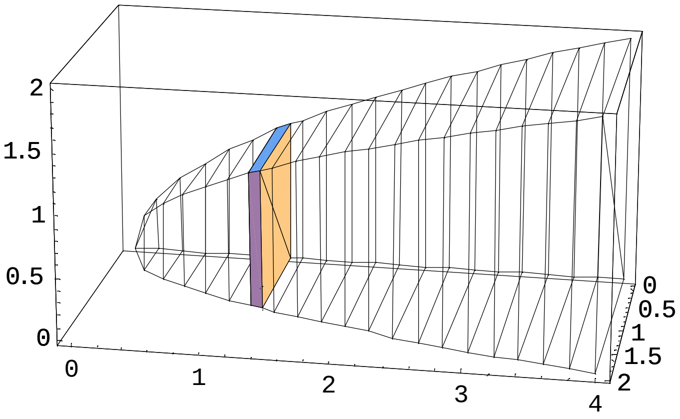
Consider the region bounded by $f(x)=\sqrt{x}$, the x-axis and $x = 4$. This region will be the base of a solid whose cross sections perpendicular to the x-axis are squares. You have to imagine the x-y coordinate plane lying flat. If you sliced this solid and looked at the end of it, you'd see a square. Below is a view of the object with some "slices" looking nearly perpendicular to the x-y plane.



The next view is looking out along the x-axis from the origin so that we can better see the square cross sections.

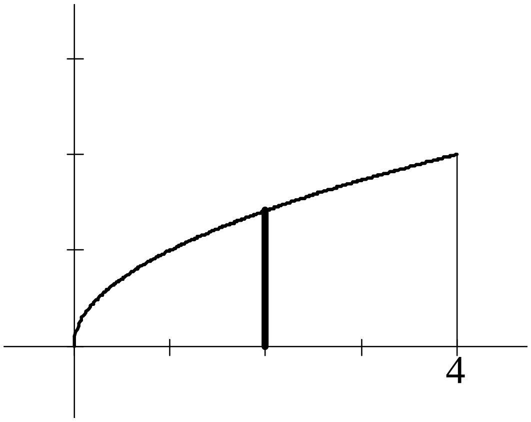


The last view below, shows an arbitrary slice within the solid. To correctly visualize this diagram, imagine the standard coordinate plane with the *y*-axis "pulled" toward you and laid down.



Now that we have some idea of what the problem looks like, let's find the volume.

First, let's find the volume of one slice—much like we found the area of one rectangle in area problems. The volume of any such slice will be the area of the cross sectional face multiplied by the height. The height for our problem is dx. Since the face is a square, its area will be the length of one side squared. On the next diagram, we are looking at our normal coordinate system and the solid would be projecting outward from the paper. The dark segment in the region represents the top view of an arbitrary slice. The length of one side of this slice is the distance from the *x*-axis to the curve. This distance will be $\sqrt{x}$.



The volume of the slice will then be $\left[\sqrt{x}\right]^2 dx$, the area of the face multiplied by the height.

If we now integrate this from 0 to 4, we will obtain the volume of our solid.

$$\begin{aligned}\int_0^4 \left[\sqrt{x}\right]^2 dx &= \int_0^4 x\ dx \\ &= \frac{1}{2}x^2\Big|_0^4 \\ &= \frac{16}{2} - 0 \\ &= 8\end{aligned}$$

Therefore, the volume of our solid is 8 cubic units.

The Problem Generalized

In general to find the volume of a solid with a known cross section, we find the area of the cross sectional face, multiply this by the height—normally dx—and then integrate from the lower to the upper bound. Symbolically this is written

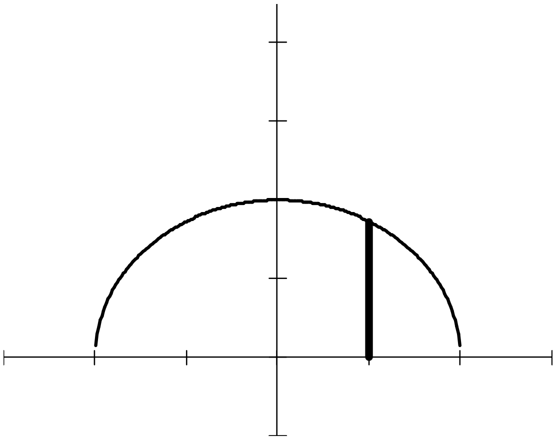
$$\int_a^b A(x)\ dx$$

where $A(x)$ is the area of the cross sectional face.

In practice, this process is usually easier than it may seem.

Example 1

Find the volume of a solid whose base region is bounded by the semicircle $f(x)=\sqrt{4-x^2}$ and the *x*-axis and whose cross section taken perpendicular to the *x*-axis is an equilateral triangle.



The slice is represented by the dark element. The solid itself is projecting outward from the paper and is an equilateral triangle. The area of an equilateral triangle is given by $\frac{\sqrt{3}}{4}(base)^2$. The base is the distance from the x-axis to the curve, $\sqrt{4-x^2}$. Since this function is a semicircle, the lower bound will be -2 and the upper bound will be 2. Our integral then becomes

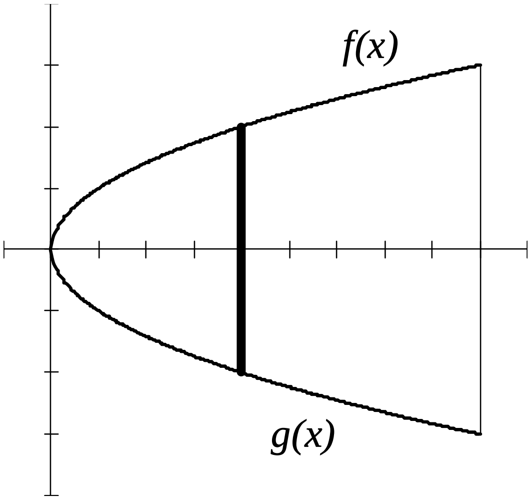
$$\begin{aligned}
V &= \int_{-2}^{2} \frac{\sqrt{3}}{4}\left[\sqrt{4-x^2}\right]^2\, dx \\
&= \frac{\sqrt{3}}{4}\int_{-2}^{2}(4-x^2)\, dx \\
&= \frac{\sqrt{3}}{4}\left(4x-\frac{1}{3}x^3\right)\Bigg|_{-2}^{2} \\
&= \frac{\sqrt{3}}{4}\left[\left(8-\tfrac{8}{3}\right)-\left(-8+\tfrac{8}{3}\right)\right] \\
&= \frac{8\sqrt{3}}{4}
\end{aligned}$$

Therefore the volume is $\frac{8\sqrt{3}}{4}$ cubic units.

Example 2

Find the volume of the solid whose base is region is bounded by $x = y^2$ and $x = 9$ and whose cross sections taken perpendicular to the x-axis are semicircles.

The curve $x = y^2$ is a parabola that opens to the right. In order to do the problem in terms of *x*, we will use two functions, one for the top half of the parabola and one for the bottom half. The top half will be $f(x) = \sqrt{x}$ and the bottom half will be $g(x) = -\sqrt{x}$. The base region with a "top" view of the slice is shown below.



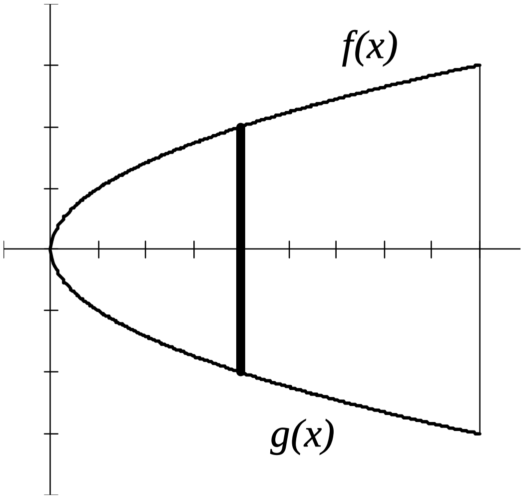
The solid is again projecting upward from the paper. If we could look straight at the cross section we would see a semicircle. The dark segment is an arbitrary slice as seen from "above".
Now, the area of a semicircle is $\frac{1}{2}p\,r^2$ and the radius of our semicircle is the distance from the *x*-axis up to *f* (or down to *g*). For this problem then, the area of the semicircle is $\frac{1}{2}p\left(\sqrt{x}\right)^2$. The thickness of the semicircle is *dx*. The volume of the solid is then given by

$$\begin{aligned}
V &= \int_0^9 \frac{1}{2}p\left(\sqrt{x}\right)^2\,dx \\
&= \frac{p}{2}\int_0^9 x\,dx \\
&= \frac{p}{2}\frac{1}{2}x^2\Big|_0^9 \\
&= \frac{p}{4}x^2\Big|_0^9 \\
&= \left[\frac{81p}{4}\right] - [0] \\
&= \frac{81p}{4}
\end{aligned}$$

Therefore the volume of the solid is $\frac{81p}{4}$ cubic units.

Example 3

Find the solid whose base region is bounded by $x = y^2$ and $x = 9$ and whose cross sections taken perpendicular to the x-axis are rectangles of height 2.



The initial setup for this problem will be the same as Example 2. The only difference is the cross section. If we could look straight at the cross section we would see a rectangle that "stuck out" 2 units from the page. The dark segment is an arbitrary slice as seen from "above".

The area of a rectangle is base times height. The base of our rectangle is the distance from *f* to *g* which is $2\sqrt{x}$. The area of the cross section then becomes $4\sqrt{x}$. The thickness is *dx* so the volume of our slice is $4\sqrt{x}\ dx$. To calculate the required volume we will evaluate

$$\begin{aligned}\int_0^9 4\sqrt{x}\ dx &= 4\int_0^9 x^{1/2}\ dx \\ &= (4)\left(\frac{2}{3}\right)x^{3/2}\Big|_0^9 \\ &= \frac{8}{3}(27) - 0 \\ &= 72\end{aligned}$$

Final note

Keep in mind that the definite integral allows us to add up an infinite number of slices. To determine the expression we want to integrate (the integrand) remember that the volume of a right solid is the area of the base times the height. A right solid is a solid with parallel sides—all of our slices are very flat right solids. The height of our solid is always either *dx* or *dy* and most of the time it will be *dx*. The area of the base is the area of the cross section in question.

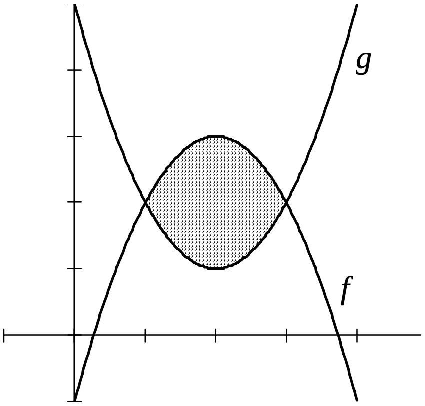
Volumes of Solids of Revolution—Disk/Washer Method

Introduction

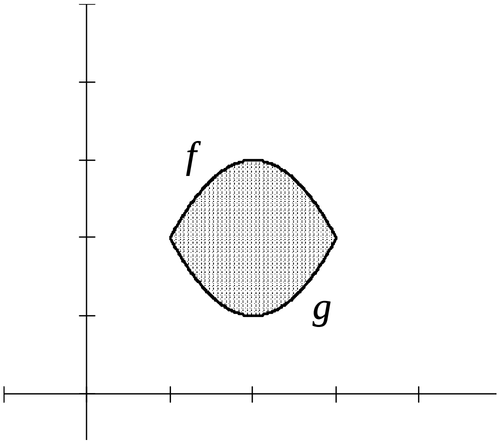
A solid of revolution is a solid generated when a particular region in the *x*-*y* plane is rotated about some horizontal or vertical line. We have two basic techniques for finding such volumes: the disk/washer method and the shell method.

Solids of revolution

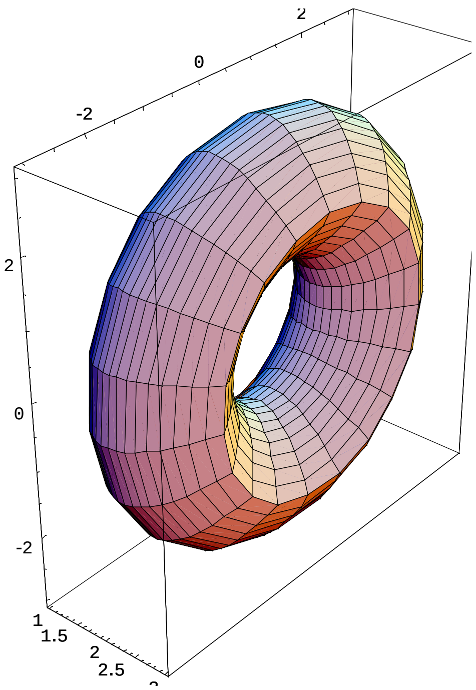
One of the biggest problems students face is visualizing solids of revolution. Although it is not necessary to be able to visualize these solids, the problem will make more sense if you can. Consider the region bounded by $f(x) = -(x-2)^2 + 3$ and $g(x) = (x-2)^2 + 1$.



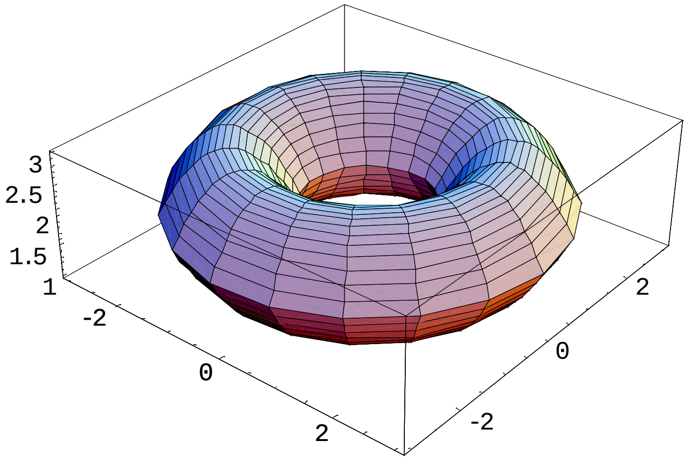
In the diagram below we have eliminated those portions of the graphs which are not in the interval $\left[1,3\right]$.



We will now rotate this region about the *x*-axis. Imagine a bar attaching the region to the *x*-axis and then rotate the region around the axis. The resulting solid is shown in the next diagram.

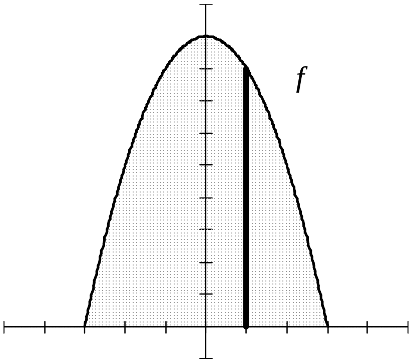


Rotating the same region about the *y*-axis would produce the solid shown below.

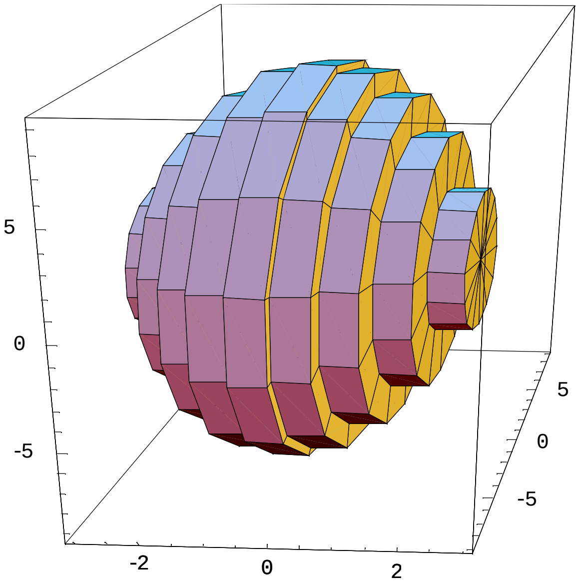


Using disks

Let's begin with a simpler function, $f(x) = 9 - x^2$, and consider the region bounded by *f* and the *x*-axis. The dark line inside the region is the element—the *i*th rectangle.



Imagine now that this region is rotated about the *x*-axis. The element would trace out a solid object we will call a "disk". The diagram below shows this solid using ten such disks. We could approximate the volume of the region using just ten disks but when we calculate the volume, we will let the number of disks go to infinity. The diagram shows that the element traces out a disk.



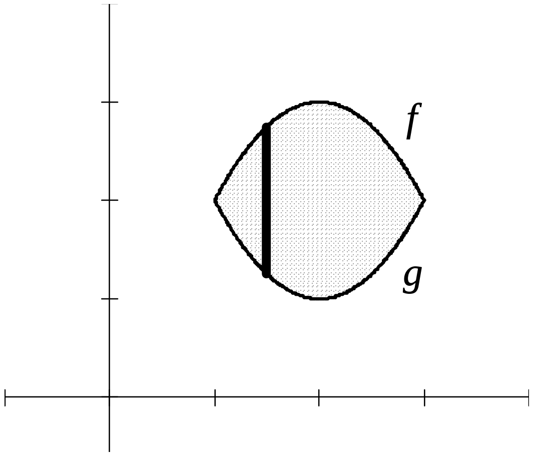
The process we use to calculate the volume of a solid is similar to the process we used to calculate the area of a region using rectangles. When we calculated area, we found an expression for the area of one arbitrary rectangle (the *i*th rectangle) and then integrated this expression from the lower to the upper bound. This integration summed an infinite number of rectangles, giving us the exact area.

To find the volume of the solid generated when the region bounded by $f(x) = 9 - x^2$ and the *x*-axis is rotated about the *x*-axis, we will find an expression for the volume of the *i*th disk and then integrate this expression from $x = -3$ to $x = 3$.

A disk is simply a very flat right cylinder. The volume of a cylinder is given by $V = p r^2 h$. In our problem, the height of the disk is *dx* and the radius of the disk is the distance from the *x*-axis to the curve. The volume of the *i*th disk is $p\left(9 - x^2\right)^2 dx$. The integral, which will yield the volume of the solid, is then $V = p\int_{-3}^{3}[9 - x^2]^2\ dx$. Evaluating this integral yields a volume of $\frac{1296p}{5}$ cubic units.

Using washers

Let's return to the region with which we began the lesson—the region bounded by $f(x) = -(x-2)^2 + 3$ and $g(x) = (x-2)^2 + 1$. We will calculate the volume of the solid generated when this region is rotated about the *x*-axis. The graph below shows the region with the element drawn in.



As this region is revolved about the x-axis, the element sweeps out a disk with a hole in it—a washer. The volume of a washer is the volume of a cylinder with a smaller cylinder subtracted from the center. This volume is can be written $p\,r_{outer}^2\,dx - p\,r_{inner}^2 dx$ where r_o is the outer radius and r_i is the inner radius and the height of our cylinder is *dx*. This expression can be simplified to: $p\left(r_{outer}^2 - r_{inner}^2\right)dx$.

Whenever we measure radii, we always measure from the axis of rotation. For our problem, the outer radius is the distance from the *x*-axis to $f(x) = -(x-2)^2+3$ which is $-(x-2)^2+3$. The inner radius is the distance from the *x*-axis to $g(x) = (x-2)^2+1$ which is $(x-2)^2+1$. The volume of the solid then becomes

$$V = p\int_1^3 \left[\left[-(x-2)^2+3\right]^2 - \left[(x-2)^2+1\right]^2\right] dx$$

General Notes

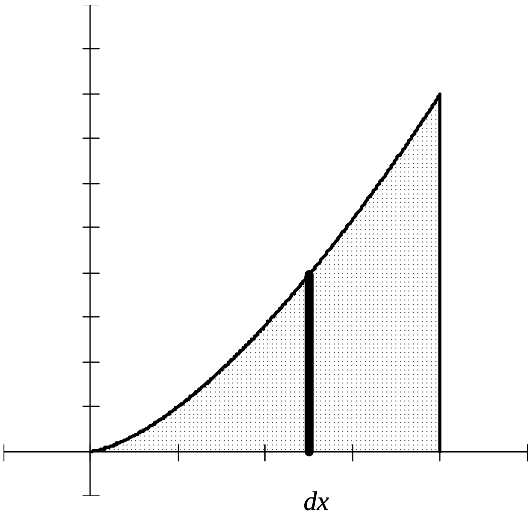
If the element is "attached" to the axis of rotation, a disk will result. If the element is not "attached" to the axis of rotation, a washer will result.

Always remember that radii are measured from the axis of rotation to the element—either to the outside of the disk, or to the outside and inside of the washer.

When we set up integrals to calculate area, we always think "(top curve minus bottom curve) times width (*dx* or *dy*)". When we use disks to calculate volume we should always think "*p* times radius squared times thickness". Using washers we think "*p* times (outer radius squared minus inner radius squared) times thickness". The "thickness" is always either *dx* or *dy* depending on how the region is being rotated.

Example 1

Find the volume of the solid generated when the region bounded by $y^2 = x^3$, $x = 4$ and the *x*-axis is rotated about the *x*-axis.



When the region is rotated about the *x*-axis, the element sweeps out a disk. Since the element is vertical, we will set up the integral in terms of *x*.

$$y^2 = x^3$$
$$y = x^{3/2}$$

We will integrate from $x = 0$ to $x = 4$.

The radius of our disk is the distance from the x-axis to the curve: $x^{3/2}$

Our integral is then

$$V = p\int_0^4 (x^{3/2})^2\ dx$$

This simplifies to

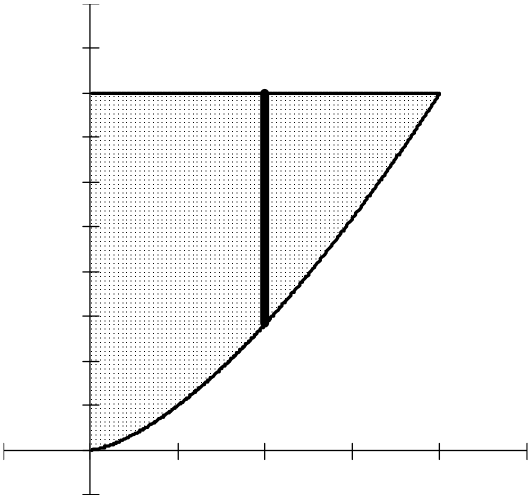
$$V = p\int_0^4 x^3\ dx$$
$$= \frac{p}{4}x^4\Big|_0^4$$
$$= 64p$$

Therefore the volume is $64p$ cubic units.

In the next several examples, we will set up the integral which will yield the necessary volume, but we will not evaluate the integrals.

Example 2

Find the volume of the solid generated when the region bounded by $y^2 = x^3$, $y = 8$ and $x = 0$ is rotated about the *x*-axis.



Again, because the element is vertical, we will set up the integral in terms of *x*.
As this region is rotated about the *x*-axis, the element sweeps out a washer so we will think "p times (outer radius squared minus inner radius squared) times thickness" to set up our integral.

The upper bound will be determined by intersecting $y = 8$ and $y = x^{3/2}$.

$$x^{3/2} = 8$$
$$x^3 = 64$$
$$x = 4$$

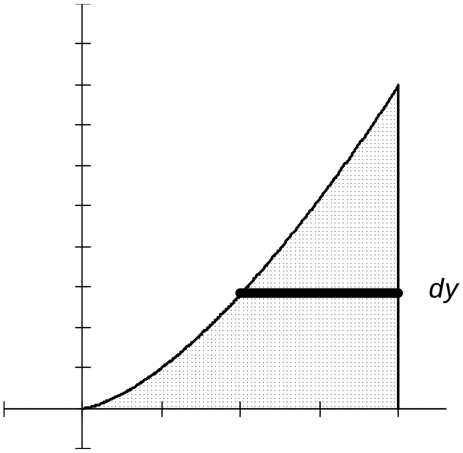
So, we will be integrating from $x = 0$ to $x = 4$.

The outer radius is the distance from the *x*-axis to the line $y = 8$, which is 8.
The inner radius is the distance from the *x*-axis to the curve $y = x^{3/2}$, which is $x^{3/2}$.
Our integral then becomes

$$V = p\int_0^4 \left[(8)^2 - \left(x^{3/2}\right)^2 \right] dx$$
$$= p\int_0^4 (64 - x^3)\ dx$$

Example 3

Find the volume of the solid generated when the region bounded by $y^2 = x^3$, $x = 4$ and the *x*-axis is rotated about the *y*-axis.



In order to generate a disk or washer, the element must be placed horizontally. When placed horizontally and rotated about the *y*-axis, the element sweeps out a washer. The means that the thickness of the washer will be ***dy*** and the must be set up in terms of *y*.

The outer radius of the washer is 4. This is the distance from the *y*-axis to the outside of the washer (the right end of our element).

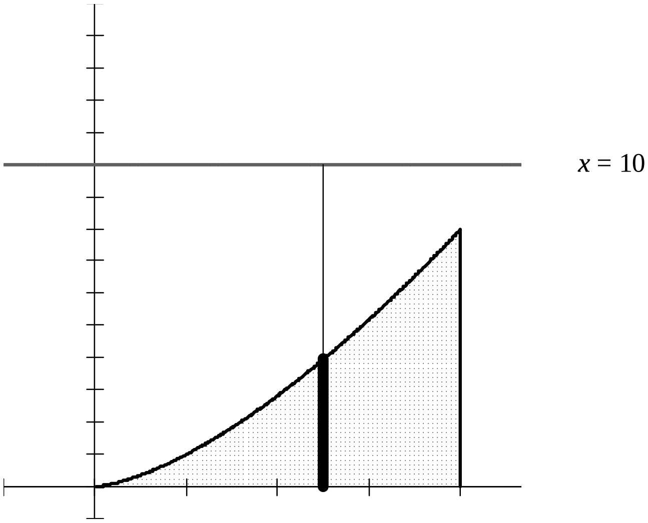
The inner radius is the distance from the *y*-axis to the curve which is $y^{2/3}$. We determined this by solving the equation $y^2 = x^3$ for *x*. Thus any point on the curve can be described by $\left(y^{2/3}, y\right)$. The left end of our element can be labeled $\left(y^{2/3}, y\right)$ and so the distance from the *y*-axis to the curve is $y^{2/3}$. Our integral then becomes

$$V = p\int_0^8 \left[(4)^2 - (y^{2/3})^2\right]\, dy$$

$$= p\int_0^8 \left[(16 - y^{2/3})^2\right]\, dy$$

Example 4

Find the volume of the solid generated when the region bounded by $y^2 = x^3$, $y = 0$ and $x = 4$ is rotated about the line $x = 10$.



To get a disk or washer when rotated about the line $x = 10$, our element must be placed vertically. A washer of height *dx* results.

The outer radius—measured from the axis of rotation to the outside of the washer—is 10.
The inner radius becomes $10 - x^{3/2}$. This is because the distance from the *x*-axis to $x = 10$ is 10 and the distance from the *x*-axis to the top of the element is $x^{3/2}$. Thus the inner radius (the dotted line) is $10 - x^{3/2}$.

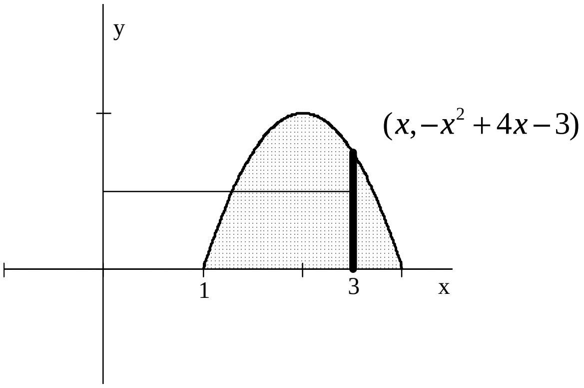
The bounds are as previously determined, $x = 0$ to $x = 4$. Our integral then becomes

$$V = p\int_0^4 \left[10^2 - (10 - x^{3/2})^2\right] dx$$

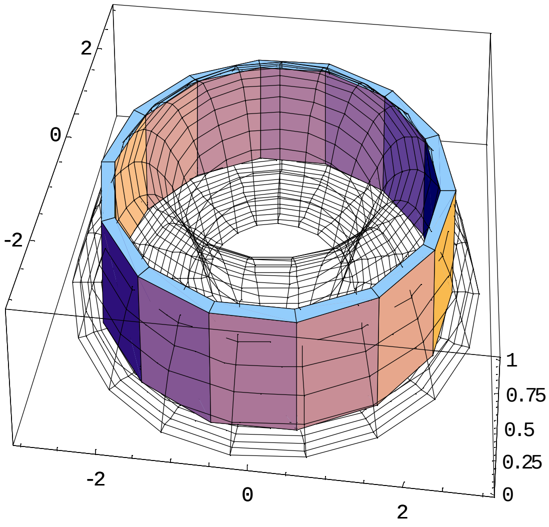
Volumes of Solids of Revolution—Shell Method

Introduction

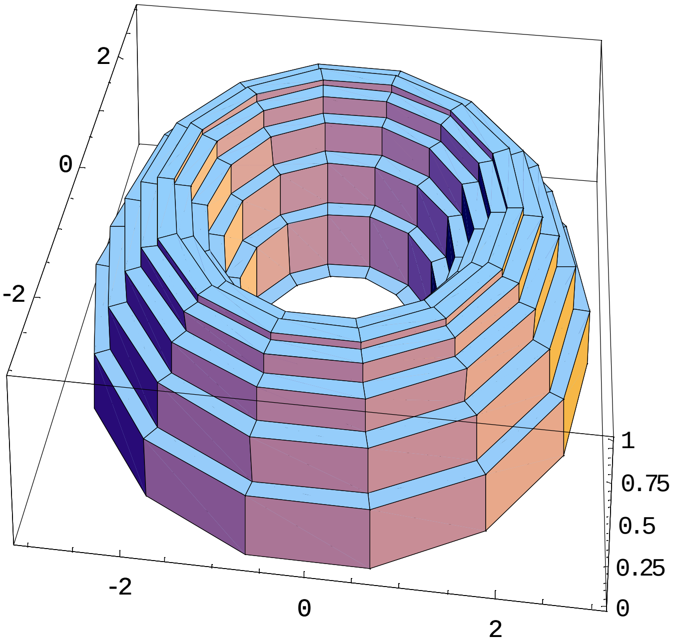
The basic difference between the disk/washer and shell methods is in the relationship between the element and the axis of rotation. You probably noticed that when you used the disk/washer method, the element was always perpendicular to the axis of rotation. If the element is drawn parallel to the axis of rotation, a shell will result. Consider the solid generated when the region bounded by $y=-x^2+4x-3$, $x=1$, $x=3$ and $y=0$ is rotated about the *y*-axis. Instead of drawing our element perpendicular to the *y*-axis so that a washer is formed, we will draw the element parallel to the axis of rotation. Below is a sketch of the region and the element.



The point where the element touches the curve can be labeled $(x,-x^2+4x-3)$. The dotted line from the *y*-axis to the element is there to represent the distance from the axis of rotation to the element (this distance is *x* in this problem). Visually, it can act as an "arm" which will swing the element about the *y*-axis. When this element is rotated around the *y*-axis, it will form a shape we call a "shell". Notice that this shell will be *dx* thick. The diagram below shows this shell (approximated with a polygon instead of a circle but you get the idea!) The wire frame is just showing the "frame" of the actual solid.



The next diagram shows an approximation of the volume using ten shells.

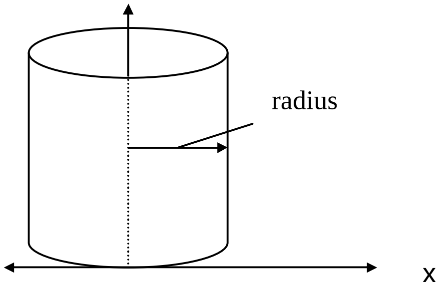


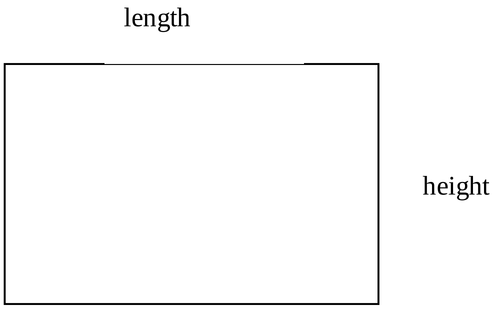
As we did with disks and washers, we will determine the volume of one arbitrary shell and then integrate that expression from $x = 1$ to $x = 3$. This integration will sum an infinite number of shells and yield the exact volume of the solid.

Finding the volume using shells

You can see that a shell is just a solid cylinder with a cylinder of slightly smaller radius subtracted from the center. There are several ways to obtain the volume. The one we will use is not usually presented in textbooks but you may find it easier to visualize.

When an element parallel to the axis of rotation is revolved, it forms a pipe-like object with a very narrow thickness—like taking a sheet of paper and forming a tube by bringing the edges of the sheet of paper together to form a tube. We need to calculate the volume of this very thin skinned "pipe". Imagine now unfolding the paper again and laying it flat. The volume of the pipe is approximately equal to the volume of the piece of paper. The volume of the sheet of paper is simply the length times the width times the height.





The length of the piece of paper is the circumference of the circle forming the top of the tube which is $2\pi r$ where *r* is the distance from the axis of rotation to the shell. (We don't have to worry about the radius going from the axis of rotation to the inside or outside of the shell—since the shell is infinitely thin, it won't matter.) The height of our piece of paper is the height of the element and the thickness is *dx* (or *dy* if our shell is horizontal).

When calculating the volume of the *i*th shell, simply think "$2\pi \cdot radius \cdot height \cdot thickness$". Once we have an expression for the volume of this arbitrary shell, we integrate from the lower to the upper bound and thus obtain the volume.

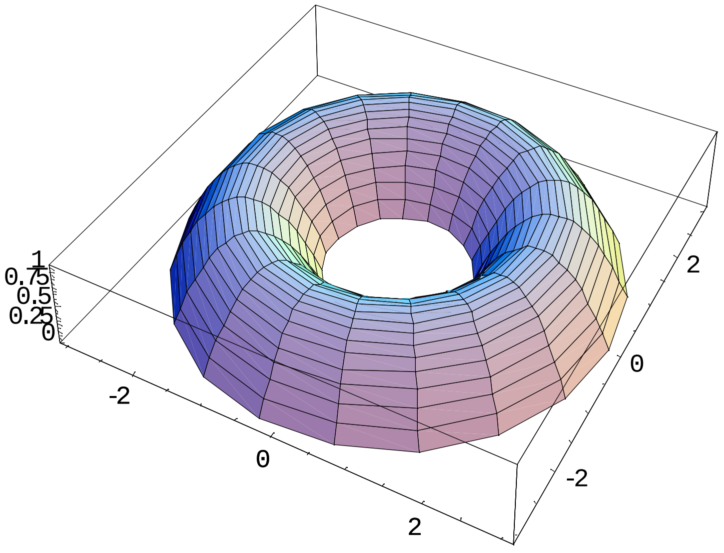
We began this discussion considering the solid generated when the region bounded by $y=-x^2+4x-3$, $x=1$, $x=3$ and $y=0$ is rotated about the *y*-axis. We will now fill in the details and calculate the volume using shells.

The radius is the distance from the axis of rotation (the *y*-axis) to the element which, in this problem, is *x*.
The height of the shell is $-x^2+4x-3$ (the distance from the bottom to the top of the shell).
The thickness of the shell is *dx*.

The volume of the shell is then $2\pi\,(x)\left(-x^2+4x-3\right)\,dx$.
The volume of the solid is then obtained by integrating this expression from $x=1$ to $x=3$.

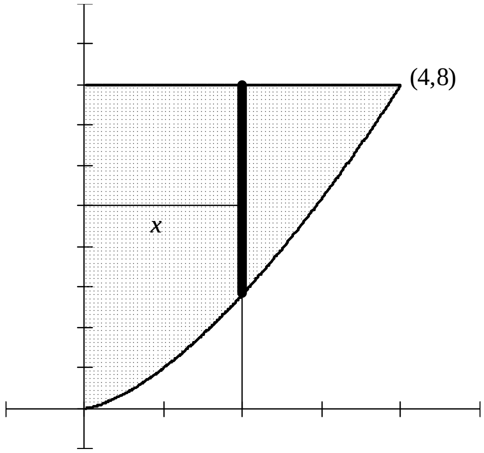
$$V=2\pi\int_1^3 (x)\left(-x^2+4x-3\right)\,dx$$

The region is illustrated below.



Example 1

Find the volume of the solid generated when the region bounded by $y^2 = x^3$, $y = 8$ and $x = 0$ is rotated about the *y*-axis.



In order to get a shell, we must draw the element parallel to the axis of revolution. Since the element is vertical, the integral will be set up in terms of *x*.

$$y^2 = x^3 \rightarrow y = x^{3/2}$$

The radius of the shell is the distance from the axis of rotation (the *y*-axis) to the element: x

The distance from the *x*-axis to the bottom of the shell is $x^{3/2}$ and the distance from the *x*-axis to the top of the shell is 8, so the height of the shell is $8 - x^{3/2}$.

The thickness is *dx*.

Now, thinking " $2p \cdot radius \cdot height \cdot thickness$ ” we obtain the following integral which will yield the volume of the solid:

$$V = 2p\int_0^4 x(8 - x^{3/2})\ dx$$

Evaluating yields:

$$V = 2p\int_0^4 x(8 - x^{3/2})\ dx$$

$$= 2p\left[8\ \frac{x^2}{2} - \frac{2}{7}x^{7/2}\right]_0^4$$

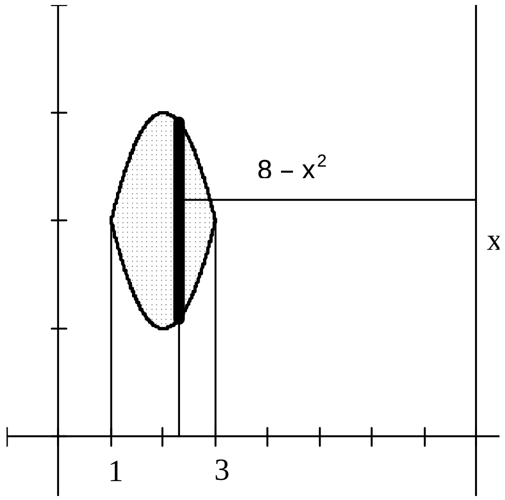
$$= 2p\left(64 - \frac{256}{7}\right) - 2p\ (0)$$

$$= \frac{384p}{7}$$

The volume is $\frac{384p}{7}$ cubic units.

Example 2

Using the shell method, find an integral which will yield the volume of the solid generated when the region bounded by $y = -(x-2)^2 + 3$ and $y = (x-2)^2 + 1$ is rotated about the line $x = 8$.



Our element is vertical so the integral will be set up in terms of *x*.

The radius of our shell is the distance from the axis of rotation ($x = 8$) to the element. The distance from the *y*-axis to the element is *x*, so the radius of the shell is $8 - x$.

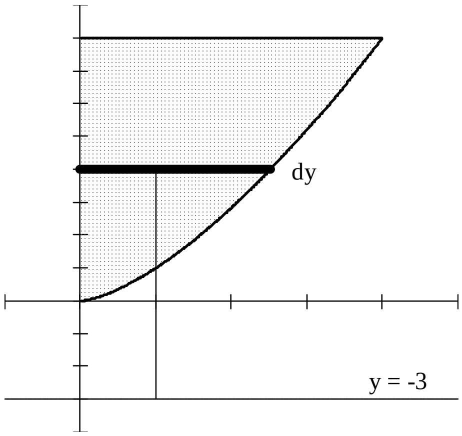
The height of our element will be $\left[-(x-2)^2 + 3\right] - \left[(x-2)^2 + 1\right]$, the difference in the two function values.

The integral then becomes

$$V = 2p\int_1^3 x[(-(x-2)^2 + 3) - ((x-2)^2 + 1)]\ dx$$

Example 3

Using the shell method, find an integral which will yield the volume of the solid generated when the region bounded by $y^2 = x^3$, $y = 8$ and $x = 0$ is rotated about the line $y = -3$.



Since the element is horizontal, the integral will be set up in terms of y.

$$y^2 = x^3 \to x = y^{2/3}$$

The right end of the element can be labeled $\left(y^{2/3}, y\right)$.

The radius of the shell is the distance from the axis of rotation, $y = -3$, to the element: $3 + y$.

The height of the element is the distance from the y-axis to the right end of the element: $y^{2/3}$.

We can now set up our integral as

$$V = 2p\int_0^8 (3 + y)(y^{2/3})\ dy$$