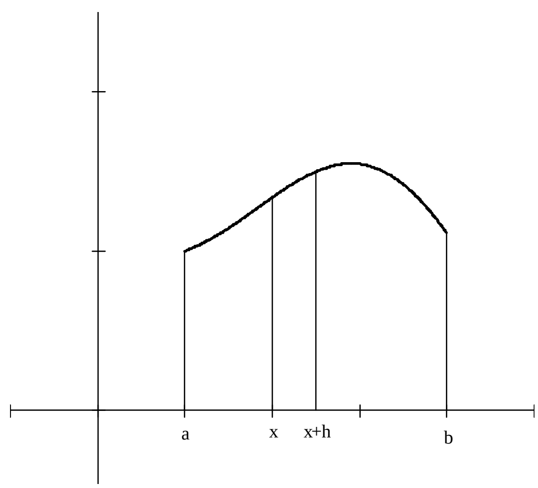
The Fundamental Theorems of Calculus

Introduction

All the pieces are in place. It's time to finally derive the Fundamental Theorems of Calculus. The Fundamental Theorems tell us several things. The First Fundamental Theorem tells us that the area under the graph of a function can be described by the antiderivative of the function and tells us how to find it. The Second Fundamental Theorem is a formal statement about the relationship between the definite integral and differentiation. Some texts list only one Fundamental Theorem (the one we are calling "The First") and explain that the second is simply a restatement of the first.

The First Fundamental Theorem of Calculus

Consider the graph of the function f below and the area under f from $x = a$ to $x = b$. The graph also shows a subinterval with x as a left endpoint and $x + h$ as a right endpoint.



Now, let's make up a function called $A(x)$ which will yield the area under f from $x = a$ to any other x value we want. We do not yet know what $A(x)$ looks like…all we know is that it will generate the area under f from $x = a$ to any other x value. Now, let's play with this function a little. Consider the following:

- $A(b)$ will give us the exact area under f from a to b.
- $A(a)$ will give us the exact area under f from a to a, which will of course be zero.
- $A(x+h)$ will give us the exact area under f from a to $x+h$.
- $A(x+h) - A(x)$ will give us the exact area of the subinterval with x as a left endpoint and $x+h$ as a right endpoint.

Now, we can say

$$f(x) \cdot h \approx A(x+h) - A(x)$$

We can say this because $f(x) \cdot h$ is the area of the left-sum rectangle in the subinterval and for a sufficiently small *h*, the area of this rectangle will be almost the same as the area of the subinterval itself.

OK, you're getting suspicious…you should be…anytime you see an expression of the form $f(x+h) - f(x)$ during a derivation you can rest assured it's not just for grins!

Let's divide both sides by *h* to get

$$f(x) \approx \frac{A(x+h) - A(x)}{h}$$

Taking a limit of both sides we can replace "$\approx$" with "$=$".

$$\lim_{h \to 0} f(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h}$$

The left side is just $f(x)$ and the right side (if you haven't noticed already) is the definition of the derivative of *A*! Therefore we can say

$$f(x) = A'(x)$$

This statement is telling us that $A(x)$ is the antiderivative of $f(x)$! If $f(x) = A'(x)$, then $\int f(x)\,dx = A(x)$. Remember, we had no idea what $A(x)$ would look like. We just made up a function which would yield the exact area under *f* from *a* to any *x* and called it *A(x)*. And now we find out that $A(x)$ is the antiderivative of $f(x)$! This means that the area under the graph of a function can be described by the antiderivative of the function itself!

Normally we denote the antiderivative of *f* with *F*. Now that we know what $A(x)$ really is, we will call it by its conventional name, $F(x)$.

Well, now we know that the area under the graph of a function is determined by the function's antiderivative. We still have no easy way to perform the calculation, but we will soon!

We know that $F(b)$ is the area under *f* from *a* to *b*.
We know that $F(a)$ is zero.
$F(b) - F(a)$ would then be a way to express the area under *f* from *a* to *b*.
We also know the definition of the definite integral: $\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x$, which can be interpreted as the area under *f* from a to b. Combining these we obtain the First Fundamental Theorem of Calculus.

The First Fundamental Theorem of Calculus

$$\int_a^b f(x)\,dx = F(b) - F(a) \text{ where } F'(x) = f(x)$$

We can now evaluate definite integrals without having to find the limit of a Riemann sum!

Example 1

Evaluate $\int_2^5 x^2 \; dx$

$$\int_2^5 x^2 \; dx = \tfrac{1}{3}x^3\Big|_2^5 = \tfrac{1}{3}5^3 - \tfrac{1}{3}2^3 = \frac{117}{3}$$

Just a little easier than using the definition of definite integral!

Example 2

Find the area under $f(x) = -x^2 + 9$ from $x = 0$ to $x = 2$.

The area can be expressed by $\int_0^2 (-x^2 + 9) \; dx$.

$$\int_0^2 (-x^2 + 9) \; dx = \left[-\tfrac{1}{3}x^3 + 9x\right]_0^2$$

$$= \left[-\tfrac{1}{3}2^3 + 9(2)\right] - \left[-\tfrac{1}{3}0^3 + 9(0)\right]$$

$$= \frac{46}{3}$$

$\therefore$ the area is $\frac{46}{3}$ square units.

Actually, area problems can be very subtle and we will go into them in great detail soon. Just as a glimpse into the future, consider the next example.

Example 3

Evaluate $\int_0^{2p} \sin x \; dx$

$$\int_0^{2p} \sin x \; dx = -\cos x\Big|_0^{2p}$$

$$= [-\cos 2p] - [-\cos 0]$$

$$= [-1] - [-1]$$

$$= 0$$

Now, if we interpret the original definite integral as the area under $\sin x$ from $x = 0$ to $x = 2p$, our answer indicates zero area! Actually, this integral represents something called "net area"—more on this later.

When we combine the First Fundamental Theorem along with substitution, we must be careful to put our entire integral in terms of *u*, not just the integrand (the expression following the $\int$ symbol). For example, when we see $\int_1^3 x^2(x^3-8)^5\ dx$, the 1 and the 3 are values of *x*. If we decide to put the problem in terms of *u*, the bounds, the 1 and the 3, must also be put in terms of *u*. Consider the following example.

Example 4

Evaluate $\int_1^3 x^2(x^3-8)^5\ dx$

$$u = x^3 - 8$$
$$du = 3x^2\ dx$$
$$\frac{1}{3}du = x^2\ dx$$
$$\text{When } x = 1 \to u = -7$$
$$x = 3 \to u = 19$$

$$\begin{aligned}\int_1^3 x^2(x^3-8)^5\ dx &= \frac{1}{3}\int_{-7}^{19} u^5\ du \\ &= \frac{1}{18}u^6\Big|_{-7}^{19} \\ &= \left[\frac{1}{18}19^6\right] - \left[\frac{1}{18}(-7)^6\right] \\ &= 2,607,124\end{aligned}$$

Warning: <u>One of the most common errors students make is failing to "change the bounds".</u> If you use substitution, you must change the bounds.

The Second Fundamental Theorem of Calculus

We now return to our discussion of the fundamental theorems. Consider the following integral.

$$\int_2^x t^2\ dt$$

Notice that the function is in terms of *t* and the upper bound is *x* instead of a constant. Let's use the fundamental theorem to evaluate it.

$$\begin{aligned}\int_2^x t^2\ dt &= \frac{1}{3}t^3\Big|_2^x \\ &= \frac{1}{3}x^3 - \frac{1}{3}2^3 \\ &= \frac{1}{3}x^3 - \frac{8}{3}\end{aligned}$$

Now consider $\frac{d}{dx}\int_2^x t^2\ dt$. This is asking us to find the derivative of the integral we just evaluated.

$$\frac{d}{dx}\int_2^x t^2\ dt = \frac{d}{dx}\left[\tfrac{1}{3}x^3 - \tfrac{8}{3}\right]$$
$$= x^2$$

This is an example of the Second Fundamental Theorem of Calculus that formalizes the relationship between integration and differentiation.

The Second Fundamental Theorem of Calculus

$$\frac{d}{dx}\int_a^x f(t)\ dt = f(x) \quad \text{where } a \text{ is a constant}$$

Note, all that really happened is the integrand $f(t)$ became a function in x instead of t. Again, this theorem puts the relationship between integration and differentiation on solid ground.

Example 5

Find $\frac{d}{dx}\int_a^x e^{t^3}\ dt$

Unlike our original example that used the function $f(t) = t^2$ as the integrand, we cannot find an antiderivative of $f(t) = e^{t^3}$. There is no substitution or other technique that will work to evaluate the integral. Although we cannot find $\int e^{t^3}\ dt$, we can find $\frac{d}{dx}\int_a^x e^{t^3}\ dt$!

$$\frac{d}{dx}\int_a^x e^{t^3}\ dt = e^{x^3}$$

There is a generalized version of the Second Fundamental Theorem which is much more useful. Consider the following:

$$\frac{d}{dx}\int_{3x}^{x^3} \sin t\ dt.$$

Since we can, let's actually find $\int_{3x}^{x^3} \sin t\ dt$

$$\int_{3x}^{x^3} \sin t\ dt = -\cos t\Big|_{3x}^{x^3} = -\cos x^3 + \cos 3x$$

Now let's differentiate our result.

$$\frac{d}{dx}\left[-\cos x^3-\left(-\cos 3x\right)\right]=(\sin x^3)(3x^2)-(\sin 3x)(3)$$

Notice that this result is sin *t* at x^3 times the derivative of x^3 minus sin *t* at 3*x* times the derivative of 3*x*.

The Second Fundamental Theorem of Calculus

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(x)\ dx = f(h(x))h'(x) - f(g(x))g'(x)$$

This version of the Second Fundamental Theorem of Calculus is actually quite easy to prove.

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(x)\ dx = \frac{d}{dx}\left[F(x)\Big|_{g(x)}^{h(x)}\right],\ \text{where } F'(x)=f(x)$$

$$=\frac{d}{dx}\left[F(h(x))-F(g(x))\right]$$

$$=F'(h(x))h'(x)-F'(g(x))g'(x)\ \text{by the chain rule.}$$

Now, since $F'(x) = f(x)$,

$$= f(h(x))h'(x) - f(g(x))g'(x)$$

Example 6

Find $\frac{d}{dx}\int_{x^2}^{\sin x} \sec^5 x\ dx$

$$\frac{d}{dx}\int_{x^2}^{\sin x} \sec^5 t\ dt = \left[\sec^5(\sin x)\right]\cos x - \left[\sec^5 x^2\right]2x$$

The Trapezoid Rule

Introduction

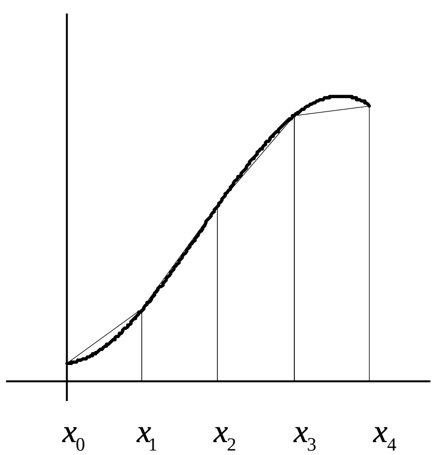
We started our recent adventures with estimating the area under a curve. Now we know that the definite integral $\int_a^b f(x)\,dx$ can give us the exact area under a curve. Apparently, when we estimated the area under a curve we were approximating the value of a definite integral! If you remember, estimating the area under a curve by adding up areas of rectangles involved no actual integration—just some multiplication and addition. Estimating the value of a definite integral without actually integrating is called "numerical" or "numeric" integration. By using only a small number of subintervals, we were performing some fairly crude numeric integrations. When mathematicians speak of numerically integrating we usually mean applying some technique (like Riemann sums—rectangles) over thousands and thousands of subintervals. Of course, if you use thousands of subintervals, your estimate is going to be pretty close to the exact value.

There are many different types of numeric integration. Up to this point, we've only used one—Riemann sums. In this section we will discuss another technique which uses the areas of trapezoids to estimate the value of a definite integral.

Keep in mind that finding areas under curves and finding the value of a definite integral are closely related but are not necessarily the same thing. We can always be asked to evaluate a definite integral, either approximately or exactly, without reference to area.

The Trapezoid Rule

The diagram below shows a function *f* on an interval $[a,b]$ where $x_0 = a$ and $x_4 = b$. The interval has been partitioned into four equal subintervals and trapezoids have been constructed inside each subinterval.



The area of a trapezoid is calculated by taking one-half the sum of the bases and multiplying by the height. The trapezoids in the diagram are "standing on end" so the height of each is Δx.

We can express the area under *f* as $\int_a^b f(x)\,dx$. Using trapezoids to approximate the value of this integral (the area under the curve in this case) we get

$$\int_a^b f(x)\,dx \approx \tfrac{1}{2}[f(x_0)+f(x_1)]\Delta x+\tfrac{1}{2}[f(x_1)+f(x_2)]\Delta x+\tfrac{1}{2}[f(x_2)+f(x_3)]\Delta x+\tfrac{1}{2}[f(x_3)+f(x_4)]\Delta x$$

This expression is simply the area of four trapezoids added together. We will now generalize this expression for *n* subintervals.

$$\int_a^b f(x)\,dx \approx \tfrac{1}{2}[f(x_0)+f(x_1)]\Delta x+\tfrac{1}{2}[f(x_1)+f(x_2)]\Delta x+\tfrac{1}{2}[f(x_2)+f(x_3)]\Delta x+\tfrac{1}{2}[f(x_3)+f(x_4)]\Delta x+\ldots$$
$$+\tfrac{1}{2}[f(x_{n-2})+f(x_{n-1})]\Delta x+\tfrac{1}{2}[f(x_{n-1})+f(x_n)]\Delta x$$

sFactoring $\tfrac{1}{2}\Delta x$ from each term yields

$$\int_a^b f(x)\,dx \approx \tfrac{1}{2}\Delta x[f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+\ldots+2f(x_{n-2})+2f(x_{n-1})+f(x_n)].$$

At this point we will use the fact that $\Delta x=\dfrac{b-a}{n}$ and make use of our summation notation to get a much more compact version of the trapezoid rule.

The Trapezoid Rule

$$\int_a^b f(x)\,dx \approx \frac{b-a}{n}\sum_{i=1}^{n} k_i\, f(x_i)$$

The k_i in the summation takes on different values depending on what *i* is. When $i=0$ or $i=n$, *k* is a 1. If *i* is any other value, *k* will be 2. This allows us to multiply each function value by the appropriate constant.

The problems involving the trapezoid rule can vary quite a bit. What we actually do to use the trapezoid rule will change for different types of problems.

Example 1

Approximate $\int_1^4 x^2\,dx$ using the trapezoid rule and 3 equal subdivisions.

This is a simple problem because we only need to use 3 subintervals. Keep in mind that all we are doing in this case is adding up the areas of three trapezoids. You can even draw a quick sketch if it helps.

$$\Delta x = \frac{4-1}{3} = 1$$
$$x_0 = 1$$
$$x_1 = 2$$
$$x_2 = 3$$
$$x_3 = 4$$

$$\int_1^4 x^2 \ dx \approx \tfrac{4-1}{6}\left[1^2 + 2(2^2) + 2(3^2) + 4^2\right]$$
$$= \tfrac{1}{2}[43]$$
$$= \tfrac{43}{2}$$

Example 2

In the chart below are a selection of *x* values and corresponding function values for a function *f*. Use the data in the table to estimate $\int_3^7 f(x)\ dx$ using the trapezoid rule with 3 subdivisions.

x	3	4.5	6.8	7
f(x)	2.8	6.9	4	1.8

The first thing you should notice is that each trapezoid is going to have its own Δx so we will not be able to factor it out. Since we are only using 3 trapezoids, it will be just as easy to write out the area of each trapezoid and then add them.

$$\int_3^7 f(x)\ dx \approx \tfrac{1}{2}[2.8+6.9](1.5) + \tfrac{1}{2}[6.9+4](2.3) + \tfrac{1}{2}[4+1.8](.2)$$
$$= 20.390$$

Example 3

Approximate $\int_1^4 x^2 \ dx$ using the trapezoid rule and 5 equal subdivisions.

When using more than 3 subintervals, a table is normally used to present all the necessary data. This is how the bulk of trapezoid rule problems should be presented.

First, $\Delta x = \tfrac{4-1}{5} = \tfrac{3}{5} = .600$

Our table will have the following columns: *i*, x_i, $f(x_i)$, k_i and $k_i f(x_i)$.

i	x_i	$f(x_i)$	k_i	$k_i\, f(x_i)$
0	1	1	1	1
1	1.600	2.560	2	5.120
2	2.200	4.840	2	9.680
3	2.800	7.840	2	15.680
4	3.400	11.560	2	23.120
5	4	16	1	16

Summing the last column yields: 70.600

Now, since $\int_a^b f(x)\,dx \approx \frac{b-a}{2n}\sum_{i=0}^{n} k_i\, f(x_i)$ we can say

$$\int_1^4 x^2\,dx \approx \tfrac{3}{10}[70.600]$$
$$= 21.180$$

Note: The $\frac{3}{10}$ came from $\frac{b-a}{2n}$ where our n is 5

Calculators and the trapezoid rule

Numeric integration lends itself to extensive use of computers and calculators. In the last example, a calculator was used to generate all but the *i* column. Lists, and the storage of lists, were also used to avoid premature rounding errors. Most numeric integration problems we will do (especially using the trapezoid rule) will be done with the TI-89. Below are the steps which were used:

Ü The x_i column was generated by entering: seq(x,x,1,4,.6)→a
- This generates a sequence of *x*'s with respect to *x*, beginning at 1, ending at 4, in increments of .6
- Because we will be using this list of *x*-values, before the "Enter" key was hit, the sequence was stored in "a".

Ü The $f(x_i)$ column was generated by entering: x^2|x = a→b
- This evaluates our function $f(x) = x^2$ at our list of *x*-values and stores the list of function values in "b".

Ü The k_i column was generated by simply creating a list: {1,2,2,2,2,1}→k
- This list was stored in "k".

Ü The $k_i\, f(x_i)$ column was generated by entering: k*b→c
- The multiplies all of our function values by the appropriate value of *k*.
- This list was stored in "c".

Ü To get the sum of the $k_i\, f(x_i)$ column, enter: sum(c)

Integration Summary

The Basics

Here is a list of the basic integration forms we have:

$$\int u^n\,du = \begin{cases} \frac{1}{n+1}u^{n+1} + C & \text{if } n \neq -1 \\ \ln|u| + C & \text{if } n = -1 \end{cases}$$

$$\int \sin u\,du = -\cos u + C$$

$$\int \cos u\,du = \sin u + C$$

$$\int \tan u\,du = \ln|\sec u| + C$$

$$\int \cot u\,du = \ln|\sin u| + C$$

$$\int \sec u\,du = \ln|\sec u + \tan u| + C$$

$$\int \csc u\,du = \ln|\csc u - \cot u| + C$$

$$\int \sec^2 u\,du = \tan u + C$$

$$\int \csc^2 u\,du = -\cot u + C$$

$$\int \sec u \tan u\,du = \sec u + C$$

$$\int \csc u \cot u\,du = -\csc u + C$$

$$\int e^u\,du = e^u + C$$

$$\int a^u\,du = \frac{a^u}{\ln a} + C$$

The Shortcuts

$$\int (Ax+B)^n\,dx = \frac{1}{A}\frac{1}{n+1}(Ax+B)^{n+1} + C \quad \text{for } n \neq -1$$

$$\int e^{Ax+B}\,dx = \frac{1}{A}e^{Ax+B} + C$$

$$\int a^{Ax+B}\,dx = \frac{a^{Ax+B}}{A\ln a} + C$$

$$\int \frac{D}{Ax+B}\,dx = \frac{D}{A}\ln|Ax+B| + C$$

$\int \cos(Ax+B)\,dx = \frac{1}{A}\sin(Ax+B) + C$ This shortcut works similarly for all the trigonometric functions when the argument is linear.

Integrals yielding the inverse trigonometric functions

There are some special integral forms which we need to be able to recognize. The following are based on the derivatives of the inverse trigonometric functions.

$$\int \frac{1}{\sqrt{a^2-u^2}}\,du = \sin^{-1}\tfrac{u}{a} + C$$

$$\int \frac{1}{a^2+u^2}\,du = \tfrac{1}{a}\tan^{-1}\tfrac{u}{a} + C$$

$$\int \frac{1}{u\sqrt{u^2-a^2}}\,du = \tfrac{1}{a}\sec^{-1}\tfrac{u}{a} + C$$

To prove any of the above theorems, take the derivative of the right side and you should get the integrand.

Example 1

Find $\int \frac{dx}{\sqrt{4-9x^2}}$

$$u = 3x$$
$$du = 3\,dx$$
$$\tfrac{1}{3}du = dx$$
$$a = 2$$

$$\int \frac{dx}{\sqrt{4-9x^2}} = \tfrac{1}{3}\int \frac{1}{\sqrt{a^2-u^2}}\,du$$
$$= \tfrac{1}{3}\sin^{-1}\tfrac{u}{a} + C$$
$$= \tfrac{1}{3}\sin^{-1}\tfrac{3x}{2} + C$$

Example 2

The inverse trigonometric functions are often the result of an integration when there is a constant in the numerator and a polynomial in the denominator.

Find $\int \frac{1}{x^2-6x+13}\,dx$

We will need to complete the square in the denominator to get the integrand into a form we can work with.

$$\int \frac{1}{x^2-6x+13}\,dx = \int \frac{1}{(x^2-6x+9)+13-9}\,dx$$
$$= \int \frac{1}{(x-3)^2+4}\,dx$$

$$u = x-3$$
$$\text{Now } du = dx$$
$$a = 2$$

$$= \int \frac{1}{u^2+a^2}\,du$$
$$= \frac{1}{a}\tan^{-1}\frac{u}{a} + C$$
$$= \frac{1}{2}\tan^{-1}\frac{x-3}{2} + C$$

We'll finish up this summary with several examples evaluating definite integrals which involve substitution.

Example 3

Evaluate $\int_0^2 x^2\sqrt{x^3+1}\,dx$

$$u = x^3+1$$
$$du = 3x^2\,dx$$
$$\frac{1}{3}du = x^2\,dx$$

$$\text{When } x = 0 \to u = 1$$
$$x = 2 \to u = 9$$

$$\int_0^2 2x^2\sqrt{x^3+1}\,dx = \frac{1}{3}\int_1^9 u^{1/2}\,du$$
$$= \frac{1}{3}\frac{2}{3}u^{3/2}\Big|_1^9$$
$$= \left[\frac{2}{9}9^{3/2}\right] - \left[\frac{2}{9}1^{3/2}\right]$$
$$= \frac{54}{9} - \frac{2}{9}$$
$$= \frac{52}{9}$$

Example 4

Evaluate $\int_0^3 x\sqrt{x+1}\ dx$

$$u = x+1 \rightarrow x = u-1$$
$$du = dx$$

When $x = 0 \rightarrow u = 1$
$x = 3 \rightarrow u = 4$

$$\begin{aligned}
\int_0^3 x\sqrt{x+1}\ dx &= \int_1^4 u^{1/2}(u-1)\ du \\
&= \int_1^4 (u^{3/2} - u^{1/2})\ du \\
&= \left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_1^4 \\
&= \left[\frac{2}{5}4^{5/2} - \frac{2}{3}4^{3/2}\right] - \left[\frac{2}{5}1^{5/2} - \frac{2}{3}1^{3/2}\right] \\
&= \left[\frac{64}{5} - \frac{16}{3}\right] - \left[\frac{2}{5} - \frac{2}{3}\right] \\
&= \frac{112}{15} - \left(-\frac{4}{15}\right) \\
&= \frac{116}{15}
\end{aligned}$$