Sigma Notation

Introduction

The definite integral (which is where we are headed) is one of the most important and fascinating concepts in Calculus. Like the derivative, it is a limit—a limit of a sum. We've got some work to do before we define it and put it to use so let's start with a review of sigma notation. Sigma notation, or summation notation, is simply an easy way for us to express a sum.

To see how the notation works, consider the following sum.

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2$$

Each term is of the form i^2. The sum begins with $i = 1$ and ends with $i = 5$, with i taking on successive <u>integer</u> values. There's no magic about the *i*, we could use any variable we wanted. It is fairly common to see *i*, *j* or *k* used. Using sigma notation, the sum is written

$$\sum_{i=1}^{5} i^2$$

The number on the underside of the sigma is the lower bound and the number on top of the sigma is the upper bound.

Why not just write out the sum? The wonderful compactness of sigma notation can be seen when we want to sum large numbers of terms. Suppose we wanted to write the sum of all the terms $i^2 + 7i$ starting at 1 and ending at 789. The first term would be 1^2 + 7(1) and the last term to be added would be 789^2 + 7(789), with 787 terms in between! Using sigma notation, we can write the sum as

$$\sum_{i=1}^{789} (i^2 + 7i)$$

Clearly, a much easier and faster way to write the sum!

Here are more examples of the notation:

$$\sum_{i=-2}^{1} (3i + 2) = [3(-2) + 2] + [3(-1) + 2] + [3(0) + 2] + [3(1) + 2]$$

$$\sum_{i=1}^{4} f(i) = f(1) + f(2) + f(3) + f(4)$$

$$\sum_{i=1}^{6} \frac{1}{i} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}$$

There are times when we do not want to specify an upper bound. In such a case, we simply use n as the upper bound. Consider the following summations:

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \ldots + (n-2) + (n-1) + n$$

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + 3^3 + \ldots + (n-2)^3 + (n-1)^3 + n^3$$

$$\sum_{i=1}^{n} A_i = A_1 + A_2 + A_3 + \ldots + A_{n-2} + A_{n-1} + A_n$$

And one more very important example…

$$\sum_{i=1}^{n} f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \ldots + f(x_{n-2})\Delta x + f(x_{n-1})\Delta x + f(x_n)\Delta x$$

Summation theorems

Theorem: $\sum_{i=1}^{n} c = cn$ (c is a constant)

Proof: Let $f(x_i) = c$, then $\sum_{i=1}^{n} c = \sum_{i=1}^{n} f(x_i)$

$$= f(x_1) + f(x_2) + f(x_3) + \ldots + f(x_{n-2}) + f(x_{n-1}) + f(x_n)$$
$$= c + c + c + \ldots + c + c + c$$
$$= nc$$

The sum of a constant is the constant times the upper bound.

Sometimes, the simplest theorems are the hardest to understand. When we let $f(x_i) = c$ where c is a constant, it does not matter what input we use, we always get c as the output. So, when the $i = 1$, the result is c, when the $i = 2$, the result is c, and so on until c is used n times.

Thus, instead of doing a sum such as $\sum_{i=1}^{4} 7$ by writing $7 + 7 + 7 + 7$ we can simply say $\sum_{i=1}^{4} 7 = 7(4) = 28$.

Theorem: $\sum_{i=1}^{n} ci = c\sum_{i=1}^{n} i$ (c is a constant)

Proof: $\sum_{i=1}^{n} cf(x_i) = cf(x_1) + cf(x_2) + cf(x_3) ... + cf(x_{n-1}) + cf(x_{n-1}) + cf(x_n)$

$$= c\left[f(x_1) + f(x_2) + f(x_3) ... + f(x_{n-1}) + f(x_{n-1}) + f(x_n) \right]$$

$$= c\sum_{i=1}^{n} f(x_i)$$

The sum of a constant times a function is the same as the constant times the sum of the function.

In other words, a constant factor can be moved "outside" the summation. For example, $\sum_{i=1}^{n} 5i^3 = 5\sum_{i=1}^{n} i^3$.

Theorem: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Proof: We can prove this theorem by writing the sum in two ways—once with the sum in ascending order and again in descending order.

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + ... + (n-2) + (n-1) + n$$

$$\sum_{i=1}^{n} i = n + (n-1) + (n-2) + ... + 3 + 2 + 1$$

We now add these two equations.

$$2\sum_{i=1}^{n} i = (n+1) + (n+1) + (n+1) + ... + (n+1) + (n+1) + (n+1)$$

$$= n(n+1)$$

Thus, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

This particular theorem has a little history behind it. Karl Friedrich Gauss, perhaps the greatest mathematician to have ever stomped upon the terra was a child prodigy. As a young student, Gauss had a teacher who, well, probably never won the "Best Liked Teacher" Award. As a punishment one day, the boys in his class were told to add all the integers from 1 to 100. They could not leave until they had the correct sum. Now, these boys didn't even have pencil and paper…they worked on a slate with a crude piece of chalk. Evidently the teacher had worked out the sum previously and watched with some enjoyment as the boys began their adding. After just a few moments, Gauss wrote down his answer and turned in his slate. The teacher was sure Gauss could not have possibly gotten the correct sum so quickly…but he was wrong. Gauss was allowed to leave. Now, Gauss didn't actually use the summation theorem as it is written above. He started to write the numbers down and saw a pattern. The sum looks like 1 + 2 + 3 +…+ 98 + 99 + 100. Gauss noticed that the first and the last numbers added to 101. So did the second and the second from last. So did the third and the third from last. Since there were 50 pairs of numbers which added to 101, the total was 50(101) or 5050. Notice that $50(101) = 50(100+1) = \frac{100(100+1)}{2}$ which is exactly what you would get if you used our theorem.

$$\sum_{i=1}^{100} i = \frac{100(100+1)}{2} \quad \text{where the n = 100}$$

The next two theorems we present without proof. The proofs can be done in much the same manner as the previous one.

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2 = \frac{n^2(n+1)^2}{4}$$

Notice that for all five theorems above, the summation always starts with ***i*** = 1. What happens if we need to sum a large number of terms and ***i*** is not 1? Consider the following sum:

$$\sum_{i=5}^{9} i = 5+6+7+8+9$$

This sum could also be written

$$\sum_{i=1}^{5} (i+4) = 5+6+7+8+9$$

Subtracting 4 from each of the bounds and then adding 4 to the argument resulted in an equivalent sum which begins with ***i*** = 1. Let's try another. Consider

$$\sum_{i=3}^{7} i^2 = 3^2+4^2+5^2+6^2+7^2$$

Subtracting 2 from each of the bounds and then adding 2 to the argument yields

$$\sum_{i=1}^{5} (i+2)^2 = 3^2+4^2+5^2+6^2+7^2$$

This process also works if we need to add to the lower bound to make it equal to 1. For example,

$$\sum_{i=-2}^{1} 3^i = 3^{-2}+3^{-1}+3^0+3^1$$

can be written

$$\sum_{i=1}^{4} 3^{i-3} = 3^{-2} + 3^{-1} + 3^{0} + 3^{1}$$

This property will be useful when we want to change the bounds on a sum—especially when we need to use one of our theorems, which all start with ***i*** = 1.

$$\sum_{i=a}^{b} f(i) = \sum_{i=a+c}^{b+c} f(i-c)$$

and

$$\sum_{i=a}^{b} f(i) = \sum_{i=a-c}^{b-c} f(i+c)$$

Example 1

Find $\sum_{i=3}^{5}(4i+3)$

Note that the lower and upper bounds are fairly close—there will be only 3 terms to sum. <u>In cases where the lower and upper bound are close, do not use any theorems, just write out the terms</u>!

$$\begin{aligned} \sum_{i=3}^{5}(4i+3) &= (12+3)+(16+3)+(20+3) \\ &= 15+19+23 \\ &= 57 \end{aligned}$$

Example 2

Find $\sum_{i=1}^{20}(5i+4)$

Unlike the previous example, this sum will have a large number of terms. The best approach on this type of problem is to do the general sum first…from ***i*** = 1 to n. Then evaluate your expression for n = 20.

$$\text{Consider } \sum_{i=1}^{n}(5i+4) = 5\frac{n(n+1)}{2} + 4n$$

Now, for n = 20, we get

$$\begin{aligned} \sum_{i=1}^{20}(5i+4) &= 5\frac{20(20+1)}{2} + 4(20) \\ &= 1130 \end{aligned}$$

Example 3

Find $\sum_{i=1}^{20} i(3i-2)$

$$\begin{aligned}
\text{Consider } \sum_{i=1}^{n} i(3i-2) &= \sum_{i=1}^{n} (3i^2-2i) \\
&= \sum_{i=1}^{n} 3i^2 - \sum_{i=1}^{n} 2i \\
&= 3\sum_{i=1}^{n} i^2 - 2\sum_{i=1}^{n} i \\
&= 3\frac{n(n+1)(2n+1)}{6} - 2\frac{n(n+1)}{2} \\
&= \frac{n(n+1)(2n+1)}{2} - n(n+1)
\end{aligned}$$

Now, for n = 20, we get

$$\begin{aligned}
\sum_{i=1}^{20} i(3i-2) &= \frac{20(20+1)(40+1)}{2} - 20(20+1) \\
&= 8190
\end{aligned}$$

Notice that in our very first step, we distributed the *i* before we used our summation theorems. We needed to do this because although the summation of a sum of terms is equivalent to the sum of the summations of the terms, the summation of a product is not equivalent to the product of the summations. Holy cow…let's try that symbolically—it'll make more sense. We distributed the *i* first because

$$\sum_{i=1}^{n} [f(i)+g(i)] = \sum_{i=1}^{n} f(i) + \sum_{i=1}^{n} g(i)$$

but

$$\sum_{i=1}^{n} f(i)g(i) \neq \sum_{i=1}^{n} f(i) \bullet \sum_{i=1}^{n} g(i).$$

In terms of our problem,

$$\sum_{i=1}^{20} i(3i-2) \neq \left(\sum_{i=1}^{20} i\right)\left(\sum_{i=1}^{20} (3i-2)\right)$$

Example 4

Find: $\sum_{i=3}^{6} \frac{2}{i(i-2)}$

Since there will be only 4 terms, we will use no theorems.

$$\sum_{i=3}^{6} \frac{2}{i(i-2)} = \frac{2}{3(1)} + \frac{2}{4(2)} + \frac{2}{5(3)} + \frac{2}{6(4)}$$

$$= \frac{17}{15}$$

Example 5 (very important example)

There will be times when we want a general expression for a sum. The ability to do this will be <u>critical</u> when we start working with the definition of the definite integral.

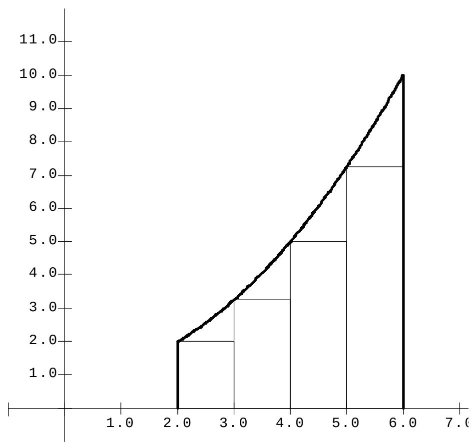
Expand and simplify: $\sum_{i=1}^{n} (3i^2 + 5i)$

$$\begin{aligned}
\sum_{i=1}^{n} (3i^2 + 5i) &= 3\frac{n(n+1)(2n+1)}{6} + 5\frac{n(n+1)}{2} \\
&= 3\frac{2n^3 + 3n^2 + n}{6} + 5\frac{n^2 + n}{2} \\
&= \frac{2n^3 + 3n^2 + n}{2} + \frac{5n^2 + 5n}{2} \\
&= \frac{2n^3 + 8n^2 + 6n}{2} \\
&= n^3 + 4n^2 + 3n
\end{aligned}$$

Area Under a Curve--Approximations

Introduction

The method we will describe here to estimate the area under a curve is an old one. In fact, it is the same method employed by Archimedes. The basic idea is simple. Since we cannot yet find the exact area under an arbitrary curve, we will estimate the area using rectangles. The graph below illustrates a problem in which four rectangles of equal width are being used to estimate the area under $f(x) = \frac{1}{4}x^2 + 1$ between $x = 2$ and $x = 6$.



To estimate the area, we add up the areas of the four rectangles. Using rectangles of equal width is not actually necessary, it's just easier. The more rectangles we use, the better our estimate. Notice that no matter how many rectangles we use, there will always be small regions on top of each rectangle that are "left out". By using increasing numbers of rectangles, we can "exhaust" the left out areas. That is why this technique is called "area by exhaustion". There are many ways that the rectangles can be drawn. In the diagram above, we used what is called a "left sum" because the left side of each rectangle is used as the height. We could have also used a right sum or midpoint sum. In addition there are *inscribed* and *circumscribed* rectangles as well. For the most part we will deal with left, right and midpoint sums.

The details

It will be best to describe the process (and the vocabulary) by using an example. We will use the following problem:

Find the area under $f(x) = -x^2 + 9$ from $x = 0$ to $x = 2$ using 4 subintervals and a left sum.

Our first step will be to *partition the interval*. To partition the interval, we divide the interval into equal parts. We partition any interval $[a,b]$ into *n* subintervals by dividing the length of the interval by *n*. Each subinterval will be Δx wide—so each of our rectangles will be Δx wide. The left end of the interval is always denoted x_o and the right end is denoted x_n. Thus $x_o = a$ and $x_n = b$. To find the xcoordinates of each subinterval, we continue adding Δx. These *x*-coordinates are collectively referred to as x_i's.

In general, to partition an interval $[a,b]$ into *n* subintervals we say:

$$
\begin{aligned}
x_o &= a \\
x_1 &= a + \Delta x \\
x_2 &= a + 2\Delta x \\
x_3 &= a + 3\Delta x \\
&\vdots \\
x_{n-2} &= a + (n-2)\Delta x \\
x_{n-1} &= a + (n-1)\Delta x \\
x_n &= b
\end{aligned}
$$

Now, when we are approximating areas, the reality of this process is much easier than it may look. To partition the interval $[0,2]$ into four subintervals we first find Δx.

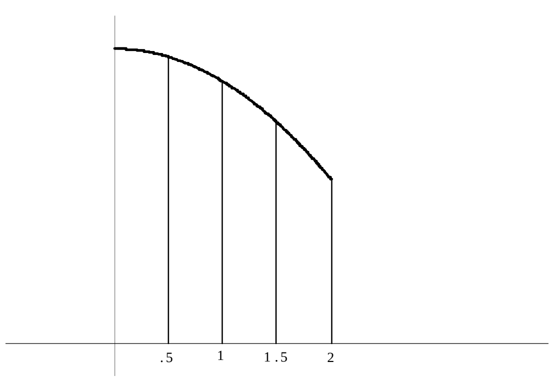
$$\Delta x = \frac{b-a}{n}$$

$$\Delta x = \frac{2-0}{4} = \frac{1}{2}$$

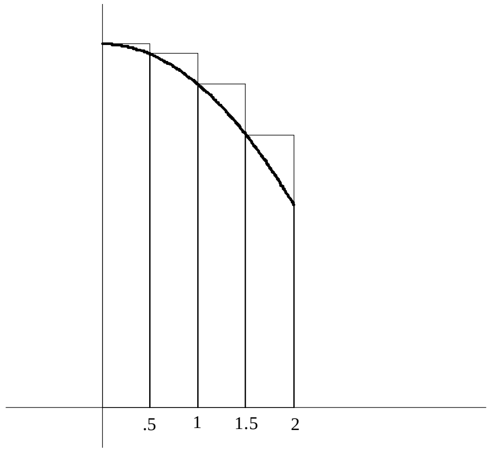
Now that we know Δx, we can proceed. We know $x_o = a$ so for us $x_o = 0$. We get the next *x*-coordinate, x_1, by adding Δx. We then get x_2 by adding Δx to x_1 and so on. The partition for our problem becomes

$$
\begin{aligned}
x_o &= 0 \\
x_1 &= \tfrac{1}{2} \\
x_2 &= 1 \\
x_3 &= \tfrac{3}{2} \\
x_4 &= 2
\end{aligned}
$$

The diagram below shows our partitioned interval.



Our problem instructs us to use a left sum. This means that we will use the value of the function at the left side of each subinterval as the height of our rectangles. The next diagram shows the rectangles for a left sum.



Now, $f(0)$ will be the height of the first rectangle, $f(.5)$ the height of the second, $f(1)$ the height of the third and $f(1.5)$ the height of the fourth. Each rectangle is Δx wide. We are now ready to estimate the area.

$$\begin{aligned} A &\approx f(x_o)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x \\ &\approx \Delta x\left[f(x_o) + f(x_1) + f(x_2) + f(x_3)\right] \\ &\approx \frac{1}{2}\left(9 + 8.750 + 8 + 6.750\right) \\ &\approx 16.250 \end{aligned}$$

Notice that we never used x_4. This is because we used a left sum. If we had used a right sum we would have used x_4 but would not have used x_0. This will always happen. When you perform a left sum, you never use the last *x*-coordinate in the interval and when you perform a right sum you never use the first *x*-coordinate in the interval.

Now let's do a right sum without all the explanation and you'll see how straightforward the process actually is.

Example 1 (right sum)

Estimate the area under $f(x) = x^2 + 5x + 6$ from $x = -1$ to $x = 5$ using 5 subintervals and a right sum.

$$\Delta x = \frac{b-a}{n} = \frac{5--1}{5} = 1.200$$

$$\begin{aligned} x_o &= -1 \\ x_1 &= .200 \\ x_2 &= 1.400 \\ x_3 &= 2.600 \\ x_4 &= 3.800 \\ x_5 &= 5 \end{aligned}$$

$$\begin{aligned} A &\approx f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x \\ &\approx \Delta x\left[f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5)\right] \\ &\approx 1.200(7.040 + 14.960 + 25.760 + 39.440 + 56) \\ &\approx 171.840 \end{aligned}$$

Example 2 (left sum)

Estimate the area under $f(x) = x^2 + 5x + 6$ from $x = -1$ to $x = 5$ using 5 subintervals and a left sum.

$$\Delta x = \frac{b-a}{n} = \frac{5--1}{5} = 1.200$$

$$\begin{aligned} x_o &= -1 \\ x_1 &= .200 \\ x_2 &= 1.400 \\ x_3 &= 2.600 \\ x_4 &= 3.800 \\ x_5 &= 5 \end{aligned}$$

$$\begin{aligned} A &\approx f(x_o)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x \\ &\approx \Delta x\left[f(x_o) + f(x_1) + f(x_2) + f(x_3) + f(x_4)\right] \\ &\approx 1.200(2 + 7.040 + 14.960 + 25.760 + 39.440) \\ &\approx 107.040 \end{aligned}$$

Example 3 (midpoint sum)

Estimate the area under $f(x)=x^2+5x+6$ from $x=-1$ to $x=5$ using 5 subintervals and a midpoint sum.

The problem starts the same—partition the interval.

$$\Delta x=\frac{b-a}{n}=\frac{5--1}{5}=1.200$$

$$\begin{aligned} x_o &= -1 \\ x_1 &= .200 \\ x_2 &= 1.400 \\ x_3 &= 2.600 \\ x_4 &= 3.800 \\ x_5 &= 5 \end{aligned}$$

We now have to find the midpoint of each subinterval.

$$m_1=\frac{x_0+x_1}{2}=-.400$$

$$m_2=\frac{x_1+x_2}{2}=.800$$

$$m_3=\frac{x_2+x_3}{2}=2$$

$$m_4=\frac{x_3+x_4}{2}=3.200$$

$$m_5=\frac{x_4+x_5}{2}=4.400$$

The midpoints can also be found by finding m_1 and then adding Δx to get m_2, and so on—which is actually quicker.

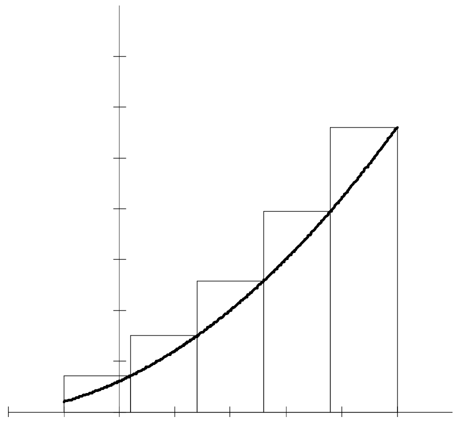
$$\begin{aligned} A &\approx f(m_1)\Delta x+f(m_2)\Delta x+f(m_3)\Delta x+f(m_4)\Delta x+f(m_5)\Delta x \\ &\approx \Delta x\left[f(m_1)+f(m_2)+f(m_3)+f(m_4)+f(m_5)\right] \\ &\approx 1.200(4.160+10.640+20+32.240+47.360) \\ &\approx 137.280 \end{aligned}$$

Notice that left, right and midpoint sums give us different approximations. That's not surprising but we should be able to put these sums in order if we are given certain information. For discussion purposes, let's denote a left sum as L_S, a right sum as R_S and a midpoint sum as M_S. If, for example, a function is always decreasing on the interval then $R_S < M_S < L_S$. If the function is always increasing on an interval, the relationship is reversed.

Example 4 (circumscribed rectangles)

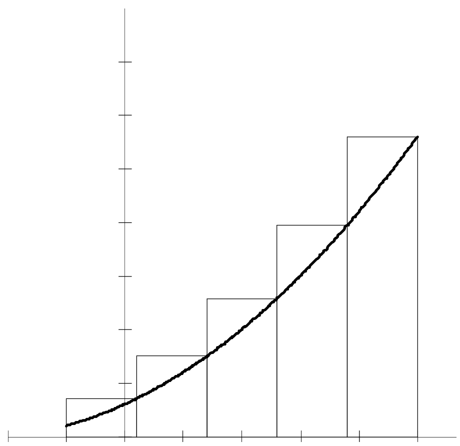
Estimate the area under $f(x) = x^2 + 5x + 6$ from $x = -1$ to $x = 5$ using 5 subintervals and circumscribed rectangles.

To use circumscribed rectangles, after we partition the interval we draw our rectangles so that the height of the rectangle is determined by the absolute maximum functin value in that interval. The diagram below shows the circumscribed rectangles.

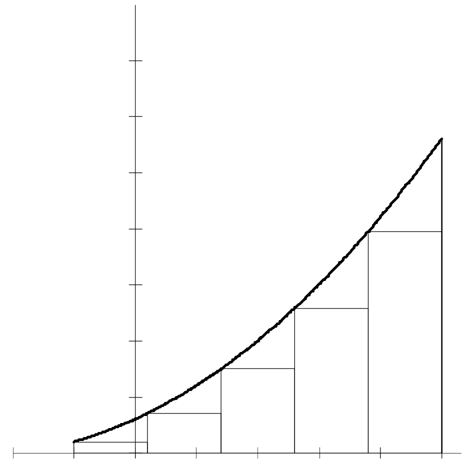


Notice that to use circumscribed rectangles on this function is equivalent to performing a right sum. The work would be exactly the same as was done in Example 1.

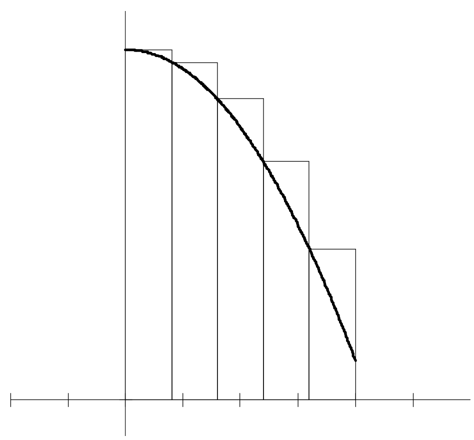
The following diagrams will help us see the relationship between inscribed/circumscribed and left/right sums for strictly increasing or strictly decreasing functions.



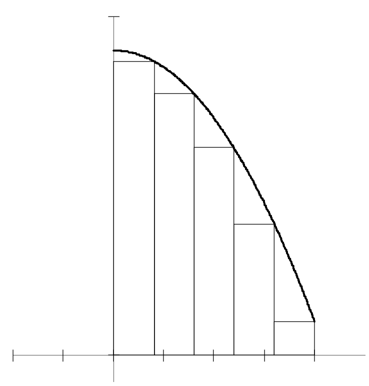
f is increasing on the interval: circumscribed = right sum



f is increasing: inscribed = left sum



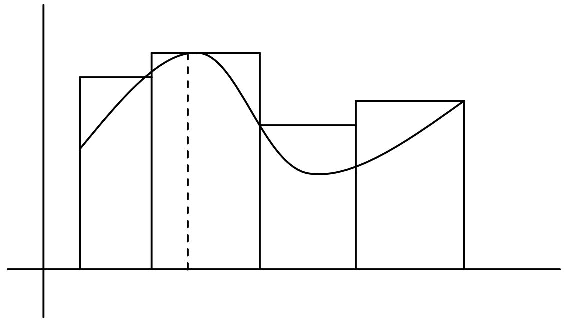
f decreasing: circumscribed = left sum



f decreasing: inscribed = right sum

If a function both increases and decreases, using circumscribed or inscribed rectangles becomes more complicated. This is why we most often use left, right or midpoint sums—it doesn't matter what the function is doing. If we are doing a left sum we always use the left side of the subinterval, if we are doing a right sum, we always use the right side of the subinterval, etc.

The diagram below illustrates circumscribed rectangles on a function which is both increasing and decreasing on the interval.



Notice that for some subintervals, the left side is used for the height of the rectangle. For other subintervals the right side is used. There's even one subinterval (the 2nd) where we would have to find the relative maximum and use it for the height.

In the end, the vast majority of the problems we face will not ask us to use circumscribed or inscribed rectangles. On the other hand, we do need to be familiar with the terminology and be able to use inscribed or circumscribed rectangles if asked to do so.

Area Under a Curve—Exact Area

Introduction

Let's return for a moment to a problem from the previous section.

Estimate the area under $f(x) = x^2 + 5x + 6$ from $x = -1$ to $x = 5$ using 5 subintervals and a right sum.

$$\Delta x = \frac{b-a}{n} = \frac{5--1}{5} = 1.200$$

$$x_o = -1,\ x_1 = .200,\ x_2 = 1.400,\ x_3 = 2.600,\ x_4 = 3.800,\ x_5 = 5$$

$$\begin{aligned} A &\approx f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x \\ &\approx \Delta x\left[f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5) \right] \\ &\approx 1.200\left(7.040 + 14.960 + 25.760 + 39.440 + 56\right) \\ &\approx 171.840 \end{aligned}$$

The sum $A \approx f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x$ could easily be expressed with sigma notation as $\sum_{i=1}^{5} f(x_i)\Delta x$. If we had performed a left sum, we would have not used x_5 but we would have used x_o and the sum would be $\sum_{i=0}^{4} f(x_i)\Delta x$. We can generalize these sums for any number of subintervals. All right sums could be written as $\sum_{i=1}^{n} f(x_i)\Delta x$ where n is the number of subintervals.

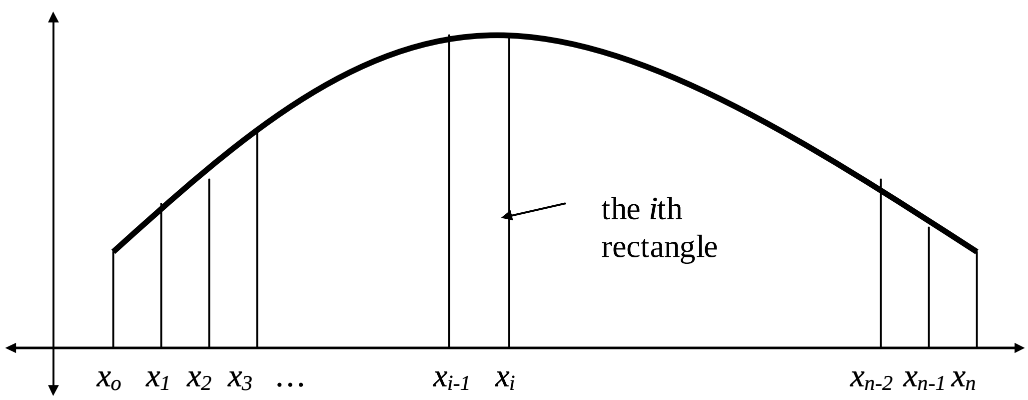
For left sums we would have $\sum_{i=0}^{n-1} f(x_i)\Delta x$. The upper bound for the left sum is $n-1$ because we do not use the right endpoint of the interval. If we had 5 subdivisions, we would use $x_o,\ x_1,\ x_2,\ x_3$ and x_4 thus the lower bound is x_o and the upper bound is x_4.

Now, the right sum $\sum_{i=1}^{n} f(x_i)\Delta x$ is in a nice form. In fact the bounds are the same as the bounds on the summation theorems we learned in the sigma notation section. The left sum $\sum_{i=0}^{n-1} f(x_i)\Delta x$ is not in a very nice form yet but we do have a theorem that will help us. Remember that $\sum_{i=a}^{b} f(i) = \sum_{i=a+c}^{b+c} f(i-c)$. This means we can say $\sum_{i=0}^{n-1} f(x_i)\Delta x = \sum_{i=1}^{n} f(x_{i-1})\Delta x$.

So, when performing a left sum on **n** subintervals we use $A=\sum_{i=1}^{n} f(x_{i-1})\Delta x$ and when performing right sums we use $A=\sum_{i=1}^{n} f(x_i)\Delta x$. Now we have to deal with partitioning an interval into **n** subdivisions. Remember that $\Delta x=\dfrac{b-a}{n}$, $x_o=a$ and $x_n=b$. Let's partition!

$$
\begin{aligned}
&x_o = a\\
&x_1 = a+\Delta x\\
&x_2 = a+2\Delta x\\
&x_3 = a+3\Delta x\\
&\vdots\\
&x_{i-1} = a+\left(i-1\right)\Delta x\\
&x_i = a+i\Delta x\\
&\vdots\\
&x_{n-2} = a+(n-2)\Delta x\\
&x_{n-1} = a+(n-1)\Delta x\\
&x_n = a+n\Delta x = b
\end{aligned}
$$

The x_{i-1} and x_i are the endpoints of something we call the *i*th rectangle. The diagram below may help.



When we find exact areas, we always use summations of left sums or right sums. We do this because it's easier than creating formulas for midpoint or other sums.

Now we're ready to find exact area. Keep in mind that $f(x_i)$ is simply the height of each rectangle and Δx is the width of each rectangle. We know that the more rectangles we use, the better our estimate of the area. If we want the exact area, we simply let the number of rectangles we use (the number of subintervals) go to infinity! This yields the following two formulas:

Exact area using a left sum: $A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\Delta x$

Exact area using a right sum: $A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$

where $\Delta x = \dfrac{b-a}{n}$ and

$$x_i = a + i\Delta x$$

$$x_{i-1} = a + (i-1)\Delta x$$

Instead of writing two different formulas for exact area, we normally write the following single formula:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x$$

where $c_i = x_i$ for right sums and $c_i = x_{i-1}$ for left sums.

You may already notice that right sums are going to be easier to perform. This is because $x_i = a + i\Delta x$ but $x_{i-1} = a + (i-1)\Delta x$. It's always going to be easier to use $a + i\Delta x$ --there'll be less algebra involved.

At this point it may all seem rather intimidating so let's run through several examples.

Example 1

Find the exact area bounded by $y = 3x,\ x = 3,\ x = 6$ and the x-axis using a right sum.

$$\Delta x = \frac{6-3}{n} = \frac{3}{n}$$

$c_i = x_i = 3 + i\Delta x$ (since we are using a right sum and a = 3)

$$A = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n} f(3+i\Delta x)\Delta x$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n} [3(3+i\Delta x)]\Delta x$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n} [9+3i\Delta x]\Delta x$$

Substituting $\Delta x = \frac{3}{n}$

$$= \lim_{n\to\infty}\sum_{i=1}^{n} \left[9+3i\frac{3}{n}\right]\frac{3}{n}$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n} \left[\frac{27}{n}+\frac{27i}{n^2}\right]$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n} \left[\frac{27}{n}+\frac{27}{n^2}i\right]$$

Remember now that i is the variable in the problem, n is a constant. This means that the $\frac{27}{n}$ is a constant. If it stands alone, the sum is simply the constant times the upper limit. If it is a coefficient of a variable, it can be moved "outside" the summation.

$$= \lim_{n\to\infty}\left[\sum_{i=1}^{n}\frac{27}{n}+\frac{27}{n^2}\sum_{I=1}^{n} i\right]$$

$$= \lim_{n\to\infty}\left[27+\frac{27}{n^2}\frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty}\left[27+\frac{27}{2}\frac{n(n+1)}{n^2}\right]$$

$$= 27+\frac{27}{2}$$

$$= \frac{81}{2}$$

$\therefore$ the area is $\frac{81}{2}$ square units.

Notice that in the third step from the bottom, where we actually took the limit as $n \to \infty$, the quotient $\frac{n(n+1)}{n^2}$ went to 1.

Example 2

Find the exact area bounded by $y = 3x,\ x = 3,\ x = 6$ and the x-axis using a left sum.

$$\Delta x = \frac{6-3}{n} = \frac{3}{n}$$

$c_i = x_{i-1} = 3 + (i-1)\Delta x$ (since we are using a left sum and a = 3)

$$
\begin{aligned}
A &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} f(3+(i-1)\Delta x)\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} [3(3+(i-1)\Delta x)]\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} [3(3+i\Delta x-\Delta x)]\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} [9+3i\Delta x-3\Delta x]\Delta x
\end{aligned}
$$

Substituting $\Delta x = \frac{3}{n}$

$$
\begin{aligned}
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left[9+\frac{9}{n}i-\frac{9}{n}\right]\frac{3}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{27}{n}+\frac{27i}{n^2}-\frac{27}{n^2}\right] \\
&= \lim_{n\to\infty} \left[\sum_{i=1}^{n}\frac{27}{n}+\frac{27}{n^2}\sum_{i=1}^{n} i-\sum_{i=1}^{n}\frac{27}{n^2}\right] \\
&= \lim_{n\to\infty} \left[27+\frac{27}{n^2}\frac{n(n+1)}{2}-\frac{27}{n}\right] \\
&= \lim_{n\to\infty} \left[27+\frac{27}{2}\frac{n(n+1)}{n^2}-\frac{27}{n}\right] \\
&= 27+\frac{27}{2}-0 \\
&= \frac{81}{2}
\end{aligned}
$$

$\therefore$ the area is $\frac{81}{2}$ square units.

Yes, the answer to the left sum and right sum are the same—after all the exact area does not change! The right sum was easier because the left sum involved more tedious algebra. This will always be true. For now, we will have to be able to do both types of sums.

There are many steps in this type of problem and many can be done at various times but here are some general guidelines:

- Find Δx first.
- Determine whether you are supposed to use $c_i = a + i\Delta x$ (right sum) or $c_i = a + (i-1)\Delta x$ (left sum).
- Remember that a is just the starting point of your interval—it's where your area begins.
- Start with $A = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x$ and replace c_i.
- Evaluate your function at $a + i\Delta x$ or $a + (i-1)\Delta x$.
- Start simplifying.
- At some point you need go replace "Δx" with its equivalent expression with the n in the denominator.
- Do the sums. Remember that summing something like $\frac{27}{n}$ is different from summing something like $\frac{27}{n}i$. Also, try to write $\frac{27}{n}i$, not $\frac{27i}{n}$.
- Find the limit.
- In at least one term, you will want to "switch" denominators…replacing an expression like $\frac{27}{n^2}\frac{n(n+1)}{2}$ with an expression that looks like $\frac{27}{2}\frac{n(n+1)}{n^2}$. This makes the limit easier to see.

Example 3

Find the exact area of the region bounded by $y = x^2 + 2$, $x = 2$, $x = 5$ and the x-axis using a right sum.

$$\Delta x = \frac{5-2}{n} = \frac{3}{n}$$
$$c_i = x_i = 2 + i\Delta x$$

$$\begin{aligned}
A &= \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x \\
&= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x \\
&= \lim_{n\to\infty}\sum_{i=1}^{n} f(2+i\Delta x)\Delta x \\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\left[(2+i\Delta x)^2 + 2\right]\Delta x \\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\left[4 + 4i\Delta x + i^2(\Delta x)^2 + 2\right]\Delta x \\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\left[4\Delta x + 4i(\Delta x)^2 + i^2(\Delta x)^3 + 2\Delta x\right]
\end{aligned}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{12}{n} + \frac{36}{n^2} i + \frac{27}{n^3} i^2 + \frac{6}{n} \right]$$

$$= \lim_{n\to\infty} \left[\sum_{i=1}^{n} \frac{12}{n} + \frac{36}{n^2} \sum_{i=1}^{n} i + \frac{27}{n^3} \sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} \frac{6}{n} \right]$$

$$= \lim_{n\to\infty} \left[12 + \frac{36}{n^2} \frac{n(n+1)}{2} + \frac{27}{n^3} \frac{n(n+1)(2n+1)}{6} + 6 \right]$$

$$= \lim_{n\to\infty} \left[12 + \frac{36}{2} \frac{n(n+1)}{n^2} + \frac{27}{6} \frac{n(n+1)(2n+1)}{n^3} + 6 \right]$$

$$= \lim_{n\to\infty} \left[12 + \frac{36}{2} \frac{n^2+n}{n^2} + \frac{27}{6} \frac{2n^3+3n^2+n}{n^3} + 6 \right]$$

$$= 12 + 18 + \frac{54}{6} + 6$$

$$= 45$$

$\therefore$ the area is 45 square units

Example 4

Find the exact area of the region bounded by $y = x^2 + 2,\ x = 2,\ x = 5$ and the x-axis using a left sum.

$$\Delta x = \frac{5-2}{n} = \frac{3}{n} \qquad c_i = x_{i-1} = a + (i-1)\Delta x = 2 + (i-1)\Delta x$$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(2 + (i-1)\Delta x)\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} [(2 + i\Delta x - \Delta x)^2 + 2]\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} [i^2(\Delta x)^2 - 2i(\Delta x)^2 + (\Delta x)^2 + 4i\Delta x - 4\Delta x + 6]\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} [i^2(\Delta x)^3 - 2i(\Delta x)^3 + (\Delta x)^3 + 4i(\Delta x)^2 - 4(\Delta x)^2 + 6\Delta x]$$

$$= \lim_{n\to\infty} \left[\sum_{i=1}^{n} \frac{27}{n^3} i^2 - \frac{54}{n^3} i + \frac{27}{n^3} + \frac{36}{n^2} i - \frac{36}{n^2} + \frac{18}{n} \right]$$

$$= \lim_{n\to\infty} \left[\frac{27}{n^3} \sum_{i=1}^{n} i^2 - \frac{54}{n^3} \sum_{i=1}^{n} i + \sum_{i=1}^{n} \frac{27}{n^3} + \frac{36}{n^2} \sum_{i=1}^{n} i - \sum_{i=1}^{n} \frac{36}{n^2} + \sum_{i=1}^{n} \frac{18}{n} \right]$$

$$= \lim_{n\to\infty} \left[\frac{27}{n^3} \frac{2n^3+3n^2+n}{6} - \frac{54}{n^3} \frac{n^2+n}{2} + \frac{27}{n^2} + \frac{36}{n^2} \frac{n^2+n}{2} + \frac{36}{n} + 18 \right]$$

$$= \lim_{n\to\infty} \left[\frac{27}{6} \frac{2n^3+3n^2+n}{n^3} - \frac{54}{2} \frac{n^2+n}{n^3} + \frac{27}{n^2} + \frac{36}{2} \frac{n^2+n}{n^2} + \frac{36}{n} + 18 \right]$$

$$= \frac{54}{6} - 0 + 0 + 18 - 0 + 18$$

$$= 45$$

The Definite Integral

Introduction

We've finally reached the point where we can define the definite integral. Once we have this definition, we will be on the brink of our ultimate goal…the Fundamental Theorems of Calculus. In the past several sections we been doing a lot of work with area under a curve. Keep in mind that this is just one of the many interpretations of the definite integral and we are using it because it's the easiest way to get at the definition.

Generalizing our Riemann sum

Let's start with the limit we've been using to calculate exact areas under curves.

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x$$

where

$$\Delta x = \frac{b-a}{n}$$

and

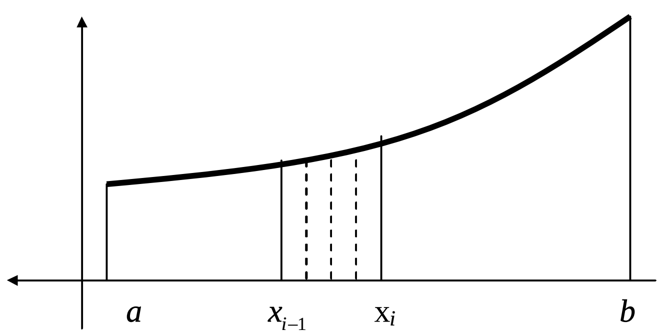
$$c_i = x_i = a + i\Delta x$$

or

$$c_i = x_{i-1} = a + (i-1)\Delta x$$

depending on whether we are doing a right or left sum. We also know that right sums are easier to do than left sums.

We realize that, in the end, if we are finding exact area, it doesn't matter whether we are doing a left sum, right sum or midpoint sum. In fact we can use any *x*-value in the interval $[x_{i-1}, x_i]$ to generate the height of the rectangle. Consider the diagram below on which only the *i*th rectangle has been drawn.



Any of the function values represented by the dashed lines could be used as the height of the rectangle. Once we let the number of rectangles go to infinity, they all become infinitely narrow and the length of the dashed lines (as well as the lengths of the lines at the endpoints) all approach the same value.

To denote that in actuality we can use any value of x in $[x_{i-1}, x_i]$ to generate the height of our rectangle we use x_i instead of c_i (which always denotes one of the endpoints of the interval). This allows us to write a more general version of our limit to find the exact area under a curve as

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x .$$

Now let's talk about the width of each subinterval. We do not really need them all to be the same width! A little mind experiment here. Picture an interval subdivided into subintervals of varying widths. Some are very narrow and some are very wide. Now, as we start to "pack" an infinite number of rectangles into the interval, they all have to shrink until they are all arbitrarily small. So it does not matter if they start out as different widths, they all get infinitely small anyway. To denote this we use $\Delta_i x$ instead of Δx. The $\Delta_i x$ tells us that each rectangle can have its own width.

In order for the limit to work with these new subintervals, we don't let the number of rectangles (subintervals) go to infinity by saying $n \to \infty$. We have a more efficient way for this to happen. The width of the largest subinterval is denoted $\|\Delta\|$ and is called "the norm". To get an infinite number of rectangles into the interval, we let the norm go to zero… $\|\Delta\| \to 0$. Now, let's once again rewrite our expression for exact area.

$$A = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(x_i)\Delta_i x$$

This is the most general form of the Riemann sum which yields the exact area under a curve. Of course, as it stands it's pretty useless in terms of actually doing problems. When we actually want to use the Riemann sum to find exact area, we will revert to the simplest form possible which is

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x .$$

Why this one? Because right sums are the easiest.

Defining the definite integral

Although $\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} f(x_i)\Delta_i x$ is a wonderfully explicit way to express the exact area under a curve, it is somewhat cumbersome. To get around the cumbersome nature of this expression, mathematicians did what mathematicians do…create a new definition complete with a new, more compact notation.

We define the definite integral as:

$$\int_a^b f(x)\ dx = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} f(x_i)\Delta_i x$$

The definite integral can be interpreted many ways and can be applied to many applications—just like the derivative. As a matter of fact, the definite integral is a mathematical object in and of itself and we will often be asked to evaluate a definite integral without reference to any particular application. As we continue moving closer to the Fundamental Theorems of Calculus however, we will continue to look at the definite integral as expressing an area under a curve.

How does this new definition change how we do our problems? It doesn't…not yet, not until we finally have the Fundamental Theorems of Calculus. Only then will our procedure for evaluating definite integrals change…and change dramatically it will! For now, we will continue to use a limit of a Riemann sum to evaluate definite integrals. Remember, right sums are always easier, so we will always use right sums to do our evaluations.

Example 1

Evaluate $\int_1^3 x^2\ dx$

$$\Delta x = \frac{3-1}{n} = \frac{2}{n}$$

Although this definite integral could represent many things, we will still look at it as the area under a curve. Since right sums are easier we will use $c_i = x_i$.

$$\begin{aligned}\int_1^3 x^2\ dx &= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x\\ &= \lim_{n\to\infty}\sum_{i=1}^{n} f(1+i\Delta x)\Delta x\\ &= \lim_{n\to\infty}\sum_{i=1}^{n}[(1+2i\Delta x+i^2(\Delta x)^2]\Delta x\end{aligned}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} [\Delta x + 2i(\Delta x)^2 + i^2(\Delta x)^3]$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{2}{n} + \frac{8}{n^2} i + \frac{8}{n^3} i^2 \right]$$

$$= \lim_{n\to\infty} \left[2 + \frac{8}{n^2} \frac{n^2+n}{2} + \frac{8}{n^3} \frac{2n^3+3n^2+n}{6} \right]$$

$$= \lim_{n\to\infty} \left[2 + \frac{8}{2} \frac{n^2+n}{n^2} + \frac{8}{6} \frac{2n^3+3n^2+n}{n^3} \right]$$

$$= 2 + 4(1) + \frac{8}{6}(2)$$

$$= \frac{26}{3}$$

$$\therefore \int_1^3 x^2 \; dx = \frac{26}{3}$$

Example 2

Evaluate $\int_5^9 (3x+2) \; dx$

$$\Delta x = \frac{9-5}{n} = \frac{4}{n} \quad \text{and} \quad x_i = 5 + i\Delta x$$

$$\int_5^9 (3x+2) \; dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(5 + i\Delta x)\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} [3(5 + i\Delta x) + 2]\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} [15 + 3i\Delta x + 2]\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} [17 + 3i\Delta x]\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} [17\Delta x + 3i(\Delta x)^2]$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[17 \frac{4}{n} + 3i \frac{16}{n^2} \right]$$

$$= \lim_{n\to\infty}\left[68 + \frac{48}{n^2}\frac{n^2+n}{2}\right]$$

$$= \lim_{n\to\infty}\left[68 + \frac{48}{2}\frac{n^2+n}{n^2}\right]$$

$$= 68 + 24$$

$$= 92$$

$$\therefore \int_5^9 (3x+2)\ dx = 92$$

Properties of the definite integral

Like any mathematical object, the definite integral has certain properties. We present them here without proof.

$$\int_a^a f(x)\ dx = 0$$

Think of this as the area under *f* from $x = a$ to $x = a$. There is no area!

$$\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx$$

This property will be more useful to us later on. It is a convenient way to switch the lower and upper bound on an integral. It will also help us with areas which lie below the *x*-axis.

$$\int_a^b k\ f(x)\ dx = k\int_a^b f(x)\ dx$$

Integrating a constant times a function is the same as integrating the function and multiplying by the constant. Constants can be "brought out in front" of the integral.

$$\int_a^b [f(x)+g(x)]\ dx = \int_a^b f(x)\ dx + \int_a^b g(x)\ dx$$

Integrating a sum (or difference) is the same as the sum (or difference) of the integrals.

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Regardless of the order of *a*, *b* and *c*.

This property will be extremely useful to us once we begin doing more involved area problems.

An example of this property would look like this: $\int_2^7 f(x)\,dx = \int_2^4 f(x)\,dx + \int_4^7 f(x)\,dx$.

If $f(x) \geq g(x)$ on $[a,b]$ then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

This property simply states that if we are looking at the graphs of two functions *f* and *g* and the graph of *f* is above the graph of *g*, then the area under *f* will be greater than the area under *g*.

$\int_a^b k\,dx = k(b-a)$ where *k* is a constant.

This is actually one definite integral we can evaluate without resorting to the definition. If *k* is a constant, we are finding the area under a horizontal line from $x = a$ to $x = b$. It's a rectangle so the area is the base ($b-a$) times the height (*k*).

Example 3

Given that $\int_{-1}^{2} x^2\,dx = 3$, find $\int_{-1}^{2} (8-x^2)\,dx$.

$$\begin{aligned}\int_{-1}^{2} (8-x^2)\,dx &= \int_{-1}^{2} 8\,dx - \int_{-1}^{2} x^2\,dx \\ &= 8(2--1)-3 \\ &= 21\end{aligned}$$

Example 4

Given that $\int_{-1}^{2} x^2\,dx = 3$ and $\int_{-1}^{2} x\,dx = \frac{3}{2}$, find $\int_{2}^{-1} 3x(x-4)\,dx$.

$$\int_{2}^{-1} 3x(x-4)\,dx = \int_{2}^{-1} (3x^2-12x)\,dx = 3\int_{2}^{-1} x^2\,dx - 12\int_{2}^{-1} x\,dx$$

$$= -3\int_{-1}^{2} x^2\,dx + 12\int_{-1}^{2} x\,dx = -3(3) + 12\left(\frac{3}{2}\right)$$

$$= 9$$