Introduction to Antidifferentiation

Introduction

Antidifferentiation is the process of finding a function whose derivative is given. Thus far in the course you have been given a function and you found the derivative. We now turn this process around…you'll be given the derivative and be asked to find the "original" function. Not to scare you, but in general the process of antidifferentiation tends to be a more subtle operation. Unlike differentiation, we will not have very many theorems to apply. Instead, antidifferentiation involves techniques rather than theorems. We will have one theorem at least, the Power Rule for Antiderivatives…which we will get to in a moment. Another important difference between differentiation and antidifferentiation is that when you differentiate a function, you get a single function as a result. When you antidifferentiate, the result is a family of functions—all of which have the same derivative.

Antidifferentiation is closely related to another process we will soon address—integration. The terms antidifferentiation and integration are sometimes used interchangeably but they are technically different—at least according to some mathematicians. The "answer" to an antidifferentiation problem is an "indefinite integral". (Yes, there is such a thing as a "definite integral" and we'll be dealing with them shortly.)

Antiderivatives

Time for a definition.

A function F is called the antiderivative of f on an interval I if
$F'(x) = f(x)\ \forall x \in I$.

The notation we use comes directly from Leibniz (whose calculus was much easier to understand than Newton's!). The definition above can be written:

$$\int f(x)dx = F(x) + C \text{ if } F'(x) = f(x)\ \forall x$$

The symbol $\int$ is actually an elongated "S" which stands for "sum". Why we use this notation will become clearer once we have a better understanding of integration. For now, think of the $\int$ and the dx as bookends. The problem $\int x^2 dx$ is asking you to find all the functions whose derivative is x^2. In general, $\int f(x)\,dx$ means "find the functions whose derivative is $f(x)$". The "+ C" on the end will be explained in a moment.

We will start with finding antiderivatives of functions which we obtained using only the Power Rule for Differentiation. We will need slightly more sophisticated techniques to handle functions which were obtained using the Chain Rule. As was mentioned above, the process of antidifferentiation (or integration) is made up more of techniques than theorems. We do have one theorem to get us started however, The Power Rule for Antiderivatives.

Before we state this Power Rule for Antiderivatives, let's look at a problem.

Let's suppose you were told that some function was differentiated and the result was $3x^2$. You need to find the function which was differentiated. Using our new notation, this problem would be stated $\int 3x^2\,dx$ .Well, when we differentiate using the Power Rule, first you multiply the function by the current exponent, then obtain a new exponent by subtracting one from the original. When we antidifferentiate, not only will we reverse the operations but we will also reverse the order in which they are done. First we will add one to the original exponent and then divide our function by this new exponent. If you add one to the exponent in $3x^2$ the new exponent becomes 3. Now we will divide the function by this new exponent and get $\frac{3x^3}{3}$ or x^3. It would seem then that the function which was differentiated was $F(x)=x^3$. (We're calling this function "$F(x)$" in order to hold to the notation in our definition of antiderivative.) There's just one problem. Although the derivative of x^3 is $3x^2$, the derivative of $x^3 + 4$ is also $3x^2$…and the derivative of $x^3 - 8$ is $3x^2$. In fact, there is an entire family of functions whose derivative is $3x^2$. To denote this, we say that the antiderivative of $3x^2$ is $x^3 + C$. Using the appropriate notation the whole problem and answer would look like this: $\int 3x^2\,dx = x^3 + C$.

Now, the process we just went through is an application of the Power Rule for Antiderivatives. To find the antiderivative of a variable raised to a power we add one to the exponent and divide by this new exponent. This isn't all of the Power Rule for Antiderivatives though…what happens if we try to antidifferentiate x^{-1}? Simply adding one to the exponent is no problem, but then we have to divide by this new exponent…which in this case is a zero! Well, we just can't do that. Remember that all we are trying to do is find the function whose derivative is x^{-1} or $\frac{1}{x}$. Anyone remember the derivative of $\ln x$? Of course! The derivative of $\ln x$ is $\frac{1}{x}$ so $\int x^{-1}\,dx = \int \frac{1}{x}dx = \ln x + C$. Again the "C" is there to signify that the derivative of $(\ln x)+\text{constant}$ is $\frac{1}{x}$. OK, we're ready for a formal statement of The Power Rule for Antiderivatives:

The Power Rule for Antiderivatives

$$\int u^n\,dx = \begin{cases} \frac{u^{n+1}}{n+1} + C \text{ for } n \neq -1 \\ \ln|u| + C \text{ for } n = -1 \end{cases}$$

We use a "*u*" instead of an "*x*" in the statement of the rule because the rule will also be applied when a function is raised to a power (which we'll do in the next section). In this statement, *u* is a function in *x*. Using the "*u*" now means we won't have to restate the rule later on. So why not use $f(x)$ like we did with our derivative theorems? We could, but things would get notationally complicated once we start dealing with composite functions…and it is more convenient and conventional to use the "u" notation.

Example 1

Find $\int x^5\,dx$.

$$\begin{aligned}\int x^5\ dx &= \frac{x^6}{6}+C\\ &= \tfrac{1}{6}x^6+C\end{aligned}$$

Example 2

Find $\int (x^2+3x-5)\,dx$.

$$\int (x^2+3x-5)\,dx = \tfrac{1}{3}x^3+\tfrac{3}{2}x^2-5x+C$$

Note: The antiderivative of a sum is the sum of the antiderivatives. Like differentiating products and quotients though, the antiderivative of a product is NOT the product of the antiderivatives. The same goes for quotients.

Also, it is pretty standard notation to use $\tfrac{1}{3}x^3+\tfrac{3}{2}x^2-5x+C$ instead of $\dfrac{x^3}{3}+\dfrac{3x^2}{2}-5x+C$, unless you have an expression with rational exponents…but either is correct and acceptable.

Example 3

Find $\int\left(\sqrt[3]{x^2}-\sqrt[3]{x}\right)\,dx$.

First rewrite the problem with rational exponents, then use the Power Rule.

$$\begin{aligned}\int\left(x^{\frac{2}{3}}-x^{\frac{1}{3}}\right)\,dx &= \tfrac{3}{5}x^{\frac{5}{3}}-\tfrac{3}{4}x^{\frac{4}{3}}+C\\ &= \tfrac{3}{5}\sqrt[3]{x^5}-\tfrac{3}{4}\sqrt[3]{x^4}+C\end{aligned}$$

Example 5

Find $\int \frac{x^2+2x}{\sqrt{x}}\,dx$.

$$\begin{aligned}\int \frac{x^2+2x}{\sqrt{x}}\,dx &= \int x^{-\frac{1}{2}}(x^2+2x)\,dx\\ &= \int\left(x^{\frac{3}{2}}+2x^{\frac{1}{2}}\right)dx\\ &= \tfrac{2}{5}x^{\frac{5}{2}}+\tfrac{4}{3}x^{\frac{3}{2}}+C\\ &= \frac{2\sqrt{x^5}}{5}+\frac{4\sqrt{x^3}}{3}+C\end{aligned}$$

Example 6

If $f''(x)=60x^4-45x^2$, find $f(x)$. (You will have two constants!)

If $f''(x)=60x^4-45x^2$,
then $f'(x)=12x^5-15x^3+C$
and $f(x)=2x^6-\frac{15}{4}x^4+Cx+D$

Now, any differentiation theorem you know can be used in reverse to find antiderivatives. For instance, you know that $D_x[\sin x]=\cos x$. This means that $\int \cos x\,dx=\sin x+C$. Be careful…right now we are restricting ourselves to functions which did not require the chain rule to differentiate. We can say that $\int \cos x\,dx=\sin x+C$ but $\int \cos(3x+5)\,dx\neq \sin(3x+5)+C$. We faced a similar situation when we were first learning derivatives. We know that $D_x[x^3]=3x^2$ but we quickly learned that $D_x[(7x-4)^3]\neq 3(7x-4)^2$. Until we learn a technique called "change of variable", or "substitution", we will be dealing with relatively simple functions. Nonetheless, we now introduce some antidifferentiation theorems which will actually always be true. The "*u*" in the theorem can stand for *x* or for a function. For now, look at it as just standing for *x*.

$$\int e^u\,du=e^u+C$$

$\int \frac{1}{u}\,du=\ln|u|+C$ (Actually part of our Power Rule. The absolute value insures we always stay in the domain of the natural logarithmic function.)

$$\int \sin u \; du = -\cos u + C$$

$$\int \cos u \; du = \sin u + C$$

$$\int \sec^2 u \; du = \tan u + C$$

$$\int \sec u \tan u \; du = \sec u + C$$

$$\int \csc u \cot u \; du = -\csc u + C$$

$$\int \csc^2 u \; du = -\cot u + C$$

All the above come directly from differentiation theorems. To prove any of the antidifferentiation theorems above, simply take the derivative of the right side and show that it is equal to the function which is to be antidifferentiated.

General and Particular Solutions

All the antiderivatives we have found so far are called indefinite integrals. They are also called general solutions—that's why the "*C*" is there—general solutions represent a family of functions. We can also find something called a "particular solution" if we are given some initial conditions. When we are looking for a particular solution, we need to determine the value of *C* which specifies the one member of the family of functions which satisfies the initial conditions.

Example 7

If $f'(x) = 12x^2 - 24x + 1$ and $f(1) = -2$, find $f(x)$.

$$\text{Since } f'(x) = 12x^2 - 24x + 1$$
$$\text{then } f(x) = 4x^3 - 12x^2 + x + C$$

$$\text{Now, because } f(1) = -2 \to -2 = 4(1)^3 - 12(1)^2 + 1 + C$$
$$C = 5$$

$$\therefore f(x) = 4x^3 - 12x^2 + x + 5$$

Example 8

If $f''(x) = 20x^3 - 10$ and $f(1) = 1$ and $f'(1) = -5$, find $f(x)$.

$$\text{Since } f''(x) = 20x^3 - 10$$
$$\text{then, } f'(x) = 5x^4 - 10x + C$$
$$\text{We know that } f'(1) = -5 \text{ so}$$
$$-5 = 5(1)^4 - 10(1) + C$$
$$C = 0$$

$$\therefore\ f\,'(x) = 5x^4 - 10x$$

Antidifferentiating yields

$$f\,(x) = x^5 - 5x^2 + D$$

Using $f\,(1) = 1$ we obtain

$$1 = (1)^5 - 5(1)^2 + D$$
$$D = 5$$

$$\therefore\ f\,(x) = x^5 - 5x^2 + 5$$

Example 9 (Very important example!)

At any point (x,y) on a curve, the slope of a tangent line is given by $4x - 5$. If the curve contains the point (3,7), find the equation of the curve.

This is our old tangent to a curve problem turned on its head. We've been given the derivative of the curve which we can antidifferentiate. This will result in a general solution—a family of curves. The given point will allow us to find the constant—and a particular solution.

We know $f\,'(x) = 4x - 5$
so, $f\,(x) = 2x^2 - 5x + C$

Since we know (3,7) is on the curve we can substitute and get

$$7 = 2(3)^2 - 5(3) + C$$
$$C = 4$$

$$\therefore\ f\,(x) = 2x^2 - 5x + 4$$

Rectilinear Motion

We have done quite a few rectilinear motion problems thus far. We now return to them knowing how to antidifferentiate. In all previous rectilinear motion problems, we were always given the position function. Now, all we will be given is some initial conditions—and we will have to come up with the velocity and position functions ourselves. Once we have the position and velocity functions, we will answer the same questions we answered with previous rectilinear motion problems…how high does the object go, how fast does it hit the ground, etc.

One item you'll need to remember—the acceleration due to gravity for a freely falling object is –32 feet per second per second. It is negative because we use up as positive and gravity pulls down. This is the starting point for most problems. If the problem involves horizontal or some other type of motion, you will be given the acceleration.

Example 10

A ball is thrown upward with a speed of 48 feet per second from the edge of a cliff 432 above the ground. Find the position function of the ball, its maximum height, the time it takes to hit the ground and its velocity at impact.

It is best to always start by listing the acceleration, velocity and position at time zero.

At $t = 0$
$a = -32$
$v = 48$
$s = 432$

Starting with acceleration…it is a constant so the acceleration function is:

$$a(t) = -32$$

Antidifferentiating,

$$v(t) = -32t + C$$

At $t = 0$, $v = 48$ so substituting to find C gives us,

$$48 = -32(0) + C$$
$$C = 48$$

Our velocity function is now

$$v(t) = -32t + 48$$

Antidifferentiating again yields,

$$s(t) = -16t^2 + 48t + D$$

At $t = 0$, $s = 432$ so we can solve for D.

$$432 = -16(0)^2 + 48(0) + D$$
$$D = 432$$

Our position function is now

$$s(t) = -16t^2 + 48t + 432$$

Now that we have the position function and the velocity function, we can answer all the usual questions. To determine how high the ball goes, we will first set the velocity equal to zero to find the time when the ball stops is upward motion. We then use this value of *t* in the position function to obtain maximum height.

$$v(t) = 0 \text{ when } -32t + 48 = 0$$
$$t = \frac{3}{2}$$

Therefore the ball will stop (at its peak) at $\frac{3}{2}$ seconds .

The maximum height will then be $s\left(\frac{3}{2}\right) = 468$

Therefore the maximum height is 468 feet.

Now, to find how long it takes to reach the ground, set the position function equal to zero.

$$s(t) = 0 \text{ when } -16t^2 + 48t + 432 = 0$$
$$t = -3.908 \text{ or } t = 6.908$$

$\therefore$ it takes approximately 6.908 seconds to hit the ground.

Now to determine the velocity at impact, find *v*(6.908)—make sure you use your calculator correctly to avoid rounding error!

v(6.908) = –173.066

$\therefore$ the velocity at impact is approximately 173.066 feet per second downward.

Example 11

A particle moves in a straight line and has acceleration given by $a(t) = 6t + 4$. If its initial velocity is –6 centimeters per second and its initial position is 9, find the position function.

Antidifferentiating the acceleration function,

$$v(t) = 3t^2 + 4t + C$$

At *t* = 0, *v* = -6 so,

$$-6 = 3(0)^2 + 4(0) + C$$
$$C = -6$$
$$\therefore v(t) = 3t^2 + 4t - 6$$

Antidifferentiating again,

$$s(t) = t^3 + 2t^2 - 6t + D$$

At t = 0, s = 9

$$9 = (0)^3 + 2(0)^2 - 6(0) + D \rightarrow D = 9$$

$$\therefore s(t) = t^3 + 2t^2 - 6t + 9$$

Antidifferentiation by Substitution

Introduction

Up to this point we have only been able to antidifferentiate relatively simple functions. Remember, when we see $\int f(x)\ dx$, $f(x)$ is the derivative of some function that we are trying to find. We will now examine a technique that will allow us to antidifferentiate functions that are the result of a more complex application of the Chain Rule. Yes, it's true that when we differentiate we always use the Chain Rule but differentiating x^3 to get $3x^2$ is a much less complicated use of the Chain Rule than when we differentiate something like $\sec^9(e^{\cos x})$. The technique we are about to learn allows us to "undo" these more involved uses of the Chain Rule.

The technique

Consider $\int (3x+2)^3\ dx$. If we tried to use our Power Rule for Antiderivatives, we would get $\frac{1}{4}(3x+2)^4+C$. Now, if this is correct, we should be able to take the derivative of our result and get $(3x+2)^3$ back. Taking the derivative of $\frac{1}{4}(3x+2)^4+C$ yields $(3x+2)^3(3)$ or $3(3x+2)^3\ldots$which is not what we wanted to get! We seem to have an "extra" factor of 3. Clearly, the Power Rule alone is not enough in this case.

Consider $\int (3x+2)^3\ dx$ again.

Now, just for fun, let's let $u = 3x + 2$.

Differentiating u would give us $\frac{du}{dx}=3$.

Multiplying both sides by dx would yield $du = 3\ dx$ or $\frac{1}{3}du=dx$.

Now, substitute u for the $3x + 2$ term and the $\frac{1}{3}du$ for the dx term. Now we have

$$\int \frac{1}{3}u^3\ du$$

which can be (and normally is) written

$$\frac{1}{3}\int u^3\ du$$

We can now antidifferentiate and get

$$\frac{1}{12}u^4 + C$$

Now replace the u with the 3x + 2 and you have

$$\frac{1}{12}(3x+2)^4 + C$$

Is this really the correct antiderivative of $(3x+2)^3$? Take the derivative of our result and you'll find out it is!

The technique just described (in great detail) is called "substitution" or "change of variable". The basic idea is to let some part of the expression you are trying to antidifferentiate be *u*, then find *du*. Everything to the right of the $\int$ symbol must be in terms of *u*. The substitution must be complete. You cannot have some *u*'s and some other variable in your problem. (Actually, you can use any letter you want—*u* is just the traditional choice.) The best way to learn this technique is to do problems. The most common question students have is, "How do I know what to let *u* be?" The answer is "Practice, practice, practice." You'll get quite good at picking an appropriate *u* after just a few problems. If your *u* doesn't work, pick something else!

Example 1

Find $\int (7x-3)^8 \ dx$.

Let $u = 7x + 3$

Now, $du = 7dx \rightarrow \frac{1}{7}du = dx$

Substituting we obtain,

$$\frac{1}{7}\int u^8 \ du$$

Now antidifferentiate,

$$\frac{1}{63}u^9 + C$$

$$\therefore \int (7x-3)^8 \ dx = \frac{1}{63}(7x-3)^9 + C$$

Example 2

Find $\int \sqrt{3x+4}\ dx$.

When we do this one, we'll show it the way it is normally shown.

$$\begin{aligned} \text{Let } u &= 3x+4 \\ du &= 3\,dx \\ \tfrac{1}{3}du &= dx \\ \int \sqrt{3x+4}\ dx &= \tfrac{1}{3}\int u^{\frac{1}{2}}\ du \\ &= \tfrac{1}{3}\tfrac{2}{3}u^{\frac{3}{2}} + C \\ &= \tfrac{2}{9}(3x+4)^{\frac{3}{2}} + C \end{aligned}$$

Example 3

Find $\int x^2(5+2x^3)^8\ dx$.

$$\begin{aligned} \text{Let } u &= 5+2x^3 \\ du &= 6x^2\ dx \\ \tfrac{1}{6}du &= x^2\ dx \\ \int x^2(5+2x^3)^8\ dx &= \tfrac{1}{6}\int u^8\ du \\ &= \tfrac{1}{6}\tfrac{1}{9}u^9 + C \\ &= \tfrac{1}{54}(5+2x^3)^9 + C \end{aligned}$$

Example 4

Find $\int x\cos x^2\ dx$.

$$\begin{aligned} \text{Let } u &= x^2 \\ du &= 2x\,dx \\ \tfrac{1}{2}du &= x\,dx \\ \int x\cos x^2\ dx &= \tfrac{1}{2}\int \cos u\ du \\ &= \tfrac{1}{2}\sin u + C \\ &= \tfrac{1}{2}\sin x^2 + C \end{aligned}$$

Example 5

Find $\int \frac{4x^2\ dx}{(1-8x^3)^4}$.

$$\begin{aligned}
\text{Let } u &= 1-8x^3 \\
du &= -24x^2\ dx \\
-\tfrac{1}{6}\,du &= 4x^2\ dx \\
\int \frac{4x^2\ dx}{(1-8x^3)^4} &= -\tfrac{1}{6}\int u^{-4}\ du \\
&= \left(-\tfrac{1}{6}\right)\left(-\tfrac{1}{3}\right)u^{-3}+C \\
&= \tfrac{1}{18}(1-8x^3)^{-3}+C \\
&= \frac{1}{18(1-8x^3)^3}+C
\end{aligned}$$

Example 6

Find $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\ dx$.

$$\begin{aligned}
\text{Let } u &= \sqrt{x} \\
du &= \frac{1}{2\sqrt{x}}\,dx \\
2\,du &= \frac{1}{\sqrt{x}}\,dx \\
\int \frac{\sin\sqrt{x}}{\sqrt{x}}\ dx &= 2\int \sin u\ du \\
&= -2\cos u + C \\
&= -2\cos\sqrt{x} + C
\end{aligned}$$

Example 7

Find $\int \sin x\sqrt{1-\cos x}\ dx$.

$$\begin{aligned}\text{Let } u &= 1-\cos x\\ du &= \sin x\ dx\\ \int \sin x\sqrt{1-\cos x}\ dx &= \int u^{\frac{1}{2}}\ du\\ &= \frac{2}{3}u^{\frac{3}{2}}+C\\ &= \frac{2}{3}\sqrt{(1-\cos x)^3}+C\end{aligned}$$

Example 8

Find $\int e^{5x}\ dx$.

$$\begin{aligned}\text{Let } u &= 5x\\ du &= 5\,dx\\ \frac{1}{5}du &= dx\\ \int e^{5x}\ dx &= \frac{1}{5}\int e^{u}\ du\\ &= \frac{1}{5}e^{5x}+C\end{aligned}$$

Note: Differentiating *e* raised to a linear term is quite simple…we've done it many times. The general theorem is $D_x[e^{ax+b}]=ae^{ax+b}$. So $D_x[e^{7x-4}]=7e^{7x-4}$. Antidifferentiating *e* raised to a linear term is equally simple. Let's derive a theorem for it:

Consider $\int e^{ax+b}\ dx$

$$\begin{aligned}\text{Let } u &= ax+b\\ du &= a\,dx\\ \frac{1}{a}du &= dx\\ \int e^{ax+b}\ dx &= \frac{1}{a}\int e^{u}\ du\\ &= \frac{1}{a}e^{u}+C\\ &= \frac{1}{a}e^{ax+b}+C\end{aligned}$$

Example 8

Find $\int e^{9x+4}\ dx$

$$\int e^{9x+4}\ dx = \frac{1}{9}e^{9x+4} + C$$

Example 10

Find $\int 2xe^{x^2}\ dx$.

The exponent on the *e* is not linear, so be careful…you need substitution to do this problem!

$$\begin{aligned} Let\ u &= x^2 \\ du &= 2x\,dx \\ \int 2xe^{x^2}\ dx &= \int e^u\ du \\ &= e^u + C \\ &= e^{x^2} + C \end{aligned}$$

We now come to another type of substitution problem. The following examples will require two substitutions. This usually occurs when our choice for *u* is clear but the degree of this factor is less than or equal to the degree of a factor that is not part of *u*.

Example 11

Find $\int x\sqrt{x+1}\ dx$.

$$\begin{aligned} \text{Let } u &= x+1 \\ du &= dx \end{aligned}$$

This allows us to replace the *x* + 1 and the *dx* but leaves us with the *x* in front of the radical.

Since we know that *u* = *x* + 1, then *x* = *u* – 1. Now we can replace the *x*.

$$\begin{aligned} \int x\sqrt{x+1}\ dx &= \int (u-1)\sqrt{u}\ du \\ &= \int u^{\frac{1}{2}}(u-1)\ du \\ &= \int (u^{\frac{3}{2}} - u^{\frac{1}{2}})\ du \\ &= \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} + C \\ &= \frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}} + C \end{aligned}$$

Example 12

Find $\int x^2\sqrt{1+x}\ dx$.

$$\text{Let } u = 1+x$$
$$du = dx$$

Since $u = 1 + x$, then $x = u - 1$ and so $x^2 = u^2 - 2u + 1$

Now we can complete our substitution.

$$\begin{aligned}\int x^2\sqrt{1+x}\ dx &= \int (u^2-2u+1)\sqrt{u}\ du\\ &= \int u^{\frac{1}{2}}(u^2-2u+1)\ du\\ &= \int (u^{\frac{5}{2}}-2u^{\frac{3}{2}}+u^{\frac{1}{2}})\ du\\ &= \frac{2}{7}u^{\frac{7}{2}}-\frac{4}{5}u^{\frac{5}{2}}+\frac{2}{3}u^{\frac{3}{2}}+C\\ &= \frac{2}{7}(1+x)^{\frac{7}{2}}-\frac{4}{5}(1+x)^{\frac{5}{2}}+\frac{2}{3}(1+x)^{\frac{3}{2}}+C\end{aligned}$$

Antidifferentiation theorems for trigonometric functions resulting from the substitution technique

Recall that we have theorems that allow us to antidifferentiate sine and cosine. We do not yet have theorems that allow us to antidifferentiate the tangent, cotangent, secant and cosecant functions. What we do have are theorems that allow us to antidifferentiate $\sec^2 u$, $\csc^2 u$, $\sec u\tan u$ and $\csc u\cot u$.
Let's start with the tangent function.
Consider $\int \tan x\ dx$.

$$\int \tan x\ dx = \int \frac{\sin x}{\cos x}\ dx$$

$$\begin{aligned}\text{Let } u &= \cos x\\ du &= -\sin x\ dx\\ -du &= \sin x\ dx\end{aligned}$$

$$\begin{aligned}\int \tan x\ dx &= \int \frac{\sin x}{\cos x}\ dx\\ &= -\int \frac{1}{u}\ du\\ &= -\ln|u|+C\\ &= \ln|u^{-1}|+C\\ &= \ln|(\cos x)^{-1}|+C\\ &= \ln|\sec x|+C\end{aligned}$$

Consider $\int \cot x \, dx$.

$$\int \cot x \, dx = \int \frac{\cos x}{\sin x} \, dx$$

$$\text{Let } u = \sin x$$
$$du = \cos x \, dx$$

$$\begin{aligned}\int \cot x \, dx &= \int \frac{\cos x}{\sin x} \, dx \\ &= \int \frac{1}{u} \, du \\ &= \ln |u| + C \\ &= \ln |\sin x| + C\end{aligned}$$

Consider $\int \sec x \, dx$.

We begin by multiplying $\sec x$ by $\dfrac{\sec x + \tan x}{\sec x + \tan x}$

$$\begin{aligned}\int \sec x \, dx &= \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} \, dx \\ &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx\end{aligned}$$

$$\text{Let } u = \sec x + \tan x$$
$$du = (\sec x \tan x + \sec^2 x) \, dx$$

$$\begin{aligned}\int \sec x \, dx &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx \\ &= \int \frac{1}{u} \, du \\ &= \ln |u| + C \\ &= \ln |\sec x + \tan x| + C\end{aligned}$$

Using a similar process,

$$\int \csc x \, dx = \ln |\csc x - \cot x| + C$$

In summary,

$$\int \sin u \ du = -\cos u + C$$

$$\int \cos u \ du = \sin u + C$$

$$\int \tan u \ du = \ln|\sec u| + C$$

$$\int \cot u \ du = \ln|\sin u| + C$$

$$\int \sec u \ du = \ln|\sec u + \tan u| + C$$

$$\int \csc u \ du = \ln|\csc u - \cot u| + C$$

Differential Equations

Introduction

A differential equation is simply an equation that contains derivatives. Solving differential equations is an extension of our work in antidifferentiating. There are many kinds of differential equations and many techniques that can be applied to a particular problem. In fact, there are entire courses at universities dedicated to the solution of this type of equation. The course taken after Calculus III in most universities is called "Differential Equations". We will only dip our toes in the waters of this complex topic. Almost all of the differential equations we will deal with are of the simplest form—first-order, separable differential equations. "First-order" means that the highest derivative in the equation is a first derivative. "Separable" means that we can separate the variables—getting all the *x*'s on one side and *y*'s on the other for example. We will see a few second-order, separable differential equations but nothing more complicated than that.

We have already seen many, many first-order, separable differential equations. Here are a few examples:
$\frac{dy}{dx}=2x$, $\frac{dy}{dx}=3x^2$, $\frac{dy}{dx}=\frac{2x^2}{3y^4}$, $2y-\frac{dy}{dx}=5x^3y$. Here is an example of a second-order, separable differential equation: $\frac{d^2y}{dx^2}=4x+3$.

Solving differential equations

To solve a differential equation we first separate the variables and then antidifferentiate both sides of the equation.

Example 1

Solve: $\frac{dy}{dx}=2x$

$$\begin{aligned}\frac{dy}{dx}&=2x\\ dy&=2x\,dx\\ \int dy&=\int 2x\,dx\\ y&=x^2+C\end{aligned}$$

Notice that $\int dy=\int 1dy=y+C$. Also, since we antidifferentiate both sides, we actually get a constant on both sides. It is convention to combine both constants into one and place it on the right side.

When we are not given any initial conditions to allow us to solve for the constant, the solution is called a “general” solution. If we are given some initial conditions and are able to find the value of the constant, the solution is called a “particular” solution.

If asked to find a general solution, we are finding a family of functions that satisfy the original differential equation. When we are given initial conditions and find a particular solution, we are finding the one particular member of the family that satisfies the original differential equation under the restrictions of the initial conditions.

Example 2

Solve $\frac{dy}{dx} = 2x$ if $y = 7$ when $x = 2$.

The solution begins the same as in Example 1.

$$\frac{dy}{dx} = 2x$$

$$dy = 2x\,dx$$

$$\int dy = \int 2x\,dx$$

$$y = x^2 + C$$

Now, we know that $y = 7$ when $x = 2$ so we can solve for C.

$$7 = 2^2 + C$$

$$C = 3$$

Therefore, $y = x^2 + 3$

Example 3

Find the general solution of: $\frac{dy}{dx} = \frac{2x^2}{3y^2}$

$$\frac{dy}{dx} = \frac{2x^2}{3y^2}$$

$$3y^2\,dy = 2x^2\,dx$$

$$\int 3y^2\;dy = \int 2x^2\;dx$$

$$y^3 = \tfrac{2}{3}x^3 + C$$

Example 4

Solve $\frac{dy}{dx} = \frac{1+x}{xy}$ if $y = -4$ when $x = 1$.

$$\frac{dy}{dx} = \frac{1+x}{xy}$$
$$y\,dy = \frac{1+x}{x}dx$$
$$y\,dy = \left(\frac{1}{x}+1\right)dx$$
$$\int y\ dy = \int\left(\frac{1}{x}+1\right)dx$$
$$\tfrac{1}{2}y^2 = [\ln|x|] + x + C$$
$$\tfrac{1}{2}y^2 = x + \ln|x| + C$$

Now, we know that y = -4 when x = 1 so we can solve for C.

$$\frac{1}{2}(-4)^2 = 1 + \ln|1| + C$$
$$C = 7$$

Therefore,

$$\frac{1}{2}y^2 = 7 + x + \ln|x|$$

or better still...

$$y^2 = 14 + 2x + 2\ln|x|$$

Example 5

Solve: $\frac{du}{dt} = e^{u+2t}$

$$\frac{du}{dt} = e^{u+2t}$$
$$\frac{du}{dt} = e^u e^{2t}$$
$$\frac{1}{e^u}du = e^{2t}\,dt$$
$$e^{-u}\,du = e^{2t}\,dt$$

$$\int e^{-u}\,du = \int e^{2t}\,dt$$

$$-e^{-u} = \frac{1}{2}e^{2t} + C$$

$$-\frac{2}{e^{u}} = e^{2t} + D$$

Example 6

Solve: $\dfrac{dy}{dx} = \dfrac{\ln x}{xy + xy^3}$

$$\frac{dy}{dx} = \frac{\ln x}{xy + xy^3}$$

$$(xy + xy^3)\,dy = \ln x\,dx$$

$$(y + y^3)\,dy = \frac{\ln x}{x}dx$$

$$\int (y + y^3)\,dy = \int \frac{\ln x}{x}dx$$

On the right side, use substitution and let $u = \ln x \to du = \dfrac{1}{x}dx \to \int u\,du \to \tfrac{1}{2}u^2 + C$

$$\tfrac{1}{2}y^2 + \tfrac{1}{4}y^4 = \tfrac{1}{2}\ln^2 x + C$$

$$2y^2 + y^4 = 2\ln^2 x + D$$

Example 7

Solve: $\dfrac{d^2y}{dx^2} = 4x + 3$

This is a second-order differential equation. The symbol $\dfrac{d^2y}{dx^2}$ cannot be separated like $\dfrac{dy}{dx}$. We can get around this by saying $\dfrac{d^2y}{dx^2} = \dfrac{dy'}{dx}$ so,

$$\frac{dy'}{dx} = 4x + 3$$

$$dy' = (4x + 3)\,dx$$

$$\int dy' = \int (4x + 3)\,dx$$

$$y' = 2x^2 + 3x + C$$

Now, we can replace y' with $\dfrac{dy}{dx}$

$$\begin{aligned}\frac{dy}{dx} &= 2x^2 + 3x + C\\ dy &= (2x^2 + 3x + C)\,dx\\ \int dy &= \int (2x^2 + 3x + C)\,dx\\ y &= \tfrac{2}{3}x^3 + \tfrac{3}{2}x^2 + Cx + D\end{aligned}$$

Many times, we will eliminate fractions from solutions to differential equations but when the *y* is isolated, we let the equation stand as it is.

If we were given some initial conditions like “ $y' = -3$ and $y = 2$ when $x = 1$”, we would stop after the first antidifferentiation and use *y*’ = -3 when *x* = 1 to find *C*. Then, after the second antidifferentiation, use *y* = 2 when *x* = 1 to find D.

Differential equations and the exponential growth model

If we had known how to solve differential equations, the exponential model would have been much easier to derive. We started our derivation with the equation

$$\frac{dy}{dt} = ky$$

We then separated the variables (although you didn’t know why we did at the time) to get

$$\frac{1}{y}dy = k\,dt$$

At this point, we tried to find an equation which, when differentiated would give us

$$\frac{1}{y}dy = k\,dt\ .$$

Now that we know how to handle differential equations like this, we could have simply antidifferentiated to get

$$\begin{aligned}\ln|y| &= kt + C\\ y &= e^{kt+C}\\ y &= e^{kt}e^{C}\\ y &= Ae^{kt}\end{aligned}\ .$$

Remember, that for most problems, the "*A*" is the initial amount present and for problems involving investing, the "*A*" is the principle.

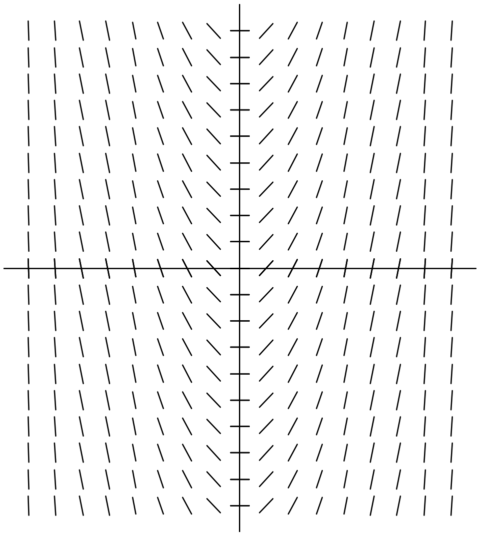
Slope Fields

Introduction

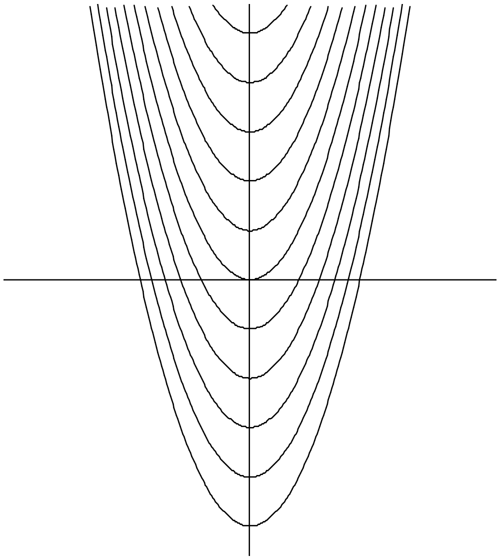
Slope fields are used to visualize the family of curves that are solutions to differential equations. Quite often we can use them to approximate solutions to differential equations. Consider the differential equation $\frac{dy}{dx} = 2x$. Solving this equation using methods we learned in the previous section, we obtain $y = x^2 + C$. This equation represents a family of parabolas and the differential equation $\frac{dy}{dx} = 2x$ tells us the slope of a tangent at any point on a member of this family. The table below shows several members of this family and the slope of a tangent for various values of *x*.

	$y = x^2$	$y = x^2 + 3$	$y = x^2 - 2$
$x = -3$	-6	-6	-6
$x = -2$	-4	-4	-4
$x = -1$	-2	-2	-2
$x = 0$	0	0	0
$x = 1$	2	2	2
$x = 2$	4	4	4
$x = 3$	3	3	3

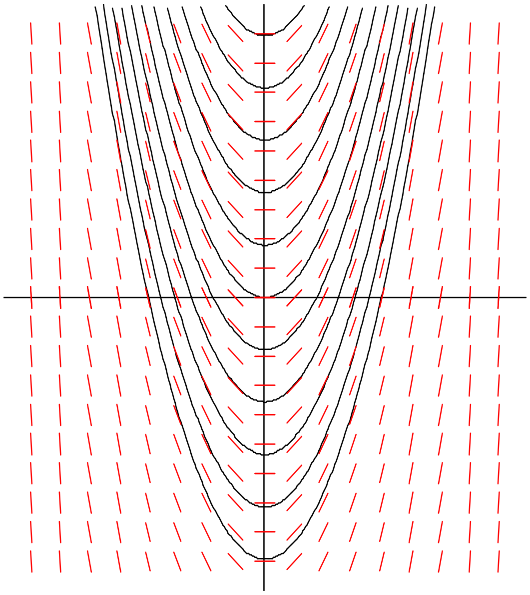
The slope field for the differential equation $\frac{dy}{dx} = 2x$ is shown below.



"Wonderful!" you say, "What is it?" Let's back up a moment and look at the graph of several members of the family of curves $y = x^2 + C$, which was the general solution to our differential equation $\frac{dy}{dx} = 2x$.



Now, let's combine the two graphs.



When we draw a slope field, we simply sketch short segments of tangent lines to selected members of the solution family. Now, the whole idea is to use a slope field to get a visual impression of the solution family before we actually solve the differential equation. Again, slope fields are a graphical representation of the general solution to a differential equation.

In our example, $\frac{dy}{dx} = 2x$, note that the slope of the tangent depends only upon *x*. This results in columns of segments that are parallel. If our differential equation were something like $\frac{dy}{dx} = 1 - y$, we would see rows of segments that would be parallel. If the differential equation involves both *x* and *y*, the slope field will have segments that are not in parallel rows and columns.

Let's walk through the procedure to draw a slope field. Consider the differential equation $\frac{dy}{dx} = -\frac{x}{y}$. In this example, the slope of a tangent to any member of the solution family depends on both *x* and *y*.

Begin by choosing a variety of points on the grid and calculating the value of $\frac{dy}{dx}$ at each point. Try to maintain some sort of orderly progress. We'll start by using coordinate pairs which all have -5 as the *x*-coordinate.

$$(-5,5) \to \frac{dy}{dx} = 1$$

$$(-5,4) \to \frac{dy}{dx} = 1.250$$

$$(-5,3) \to \frac{dy}{dx} = 1.667$$

$$(-5,2) \to \frac{dy}{dx} = 2.500$$

$$(-5,1) \to \frac{dy}{dx} = 5$$

$$(-5,0) \to \frac{dy}{dx} \nexists \text{ (vertical tangent)}$$

$$(-5,-1) \to \frac{dy}{dx} = -5$$

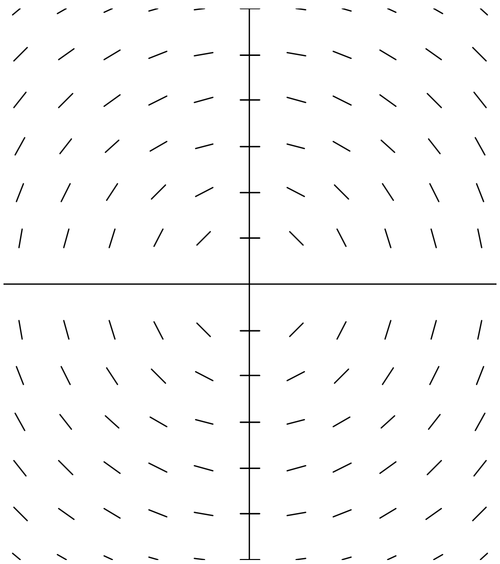
$$(-5,-2) \to \frac{dy}{dx} = -2.500$$

$$(-5,-3) \to \frac{dy}{dx} = -1.667$$

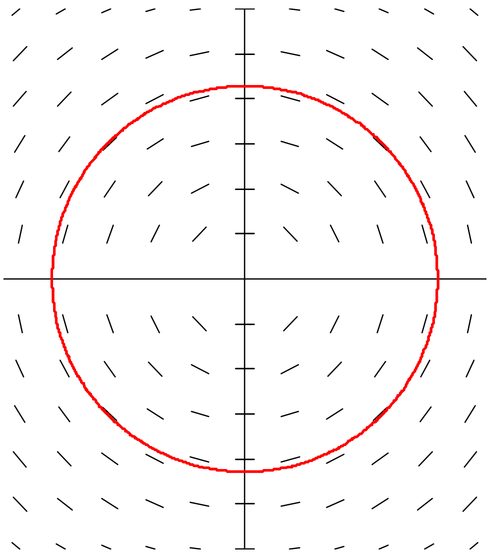
$$(-5,-4) \to \frac{dy}{dx} = -1.250$$

$$(-5,-5) \to \frac{dy}{dx} = -1$$

Now, on a blank grid, go to the point $(-5,5)$ and draw a short segment that has a slope of 1. Next, go to the point $(-5,4)$ and draw a short segment that has a slope of 1.250 and so on. Now, use –4 as the x-coordinate, calculate the value of $\frac{dy}{dx}$ and sketch the segments on your grid. When we finish, we should end up with a picture that looks like the sketch at the top of the next page.

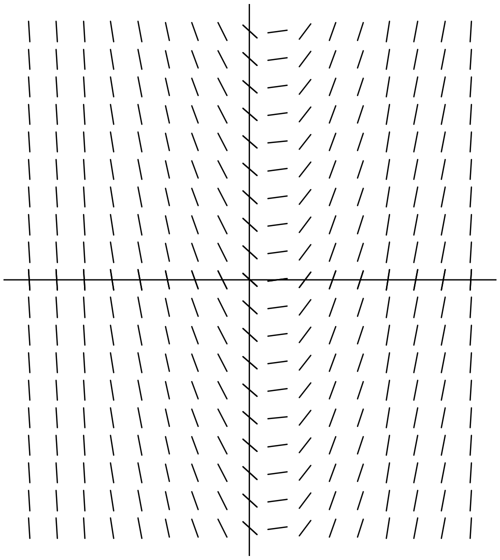


It appears that the solutions are all circles! In fact, if we wanted, we could use any initial conditions to actually draw the particular solution for the initial conditions. If the original problem stated the initial condition that when $x=4,\ y=0$ we could go to the point $(4,0)$ and follow the slope field to get the picture below.



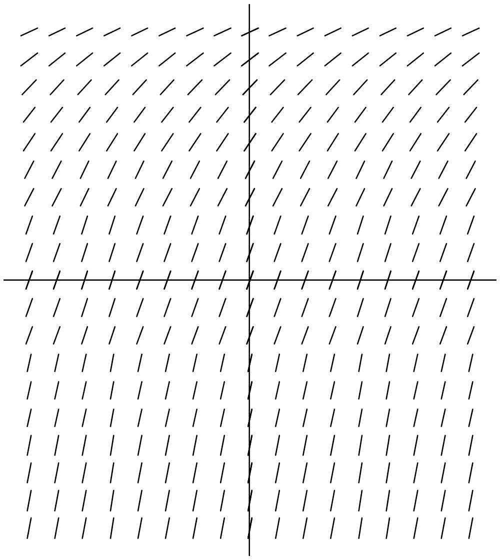
Now, what made this particular problem so tedious is that the slope depended on both *x* and *y*. If the slope depends only on *x* or *y* but not both, the process is much simpler. If the slope depends only upon *x*, we need only draw one row (say, the top row) of segments and then draw parallel segments down the columns. If the slope depends only upon *y*, we draw one column of segments (say, the left column) and then draw parallel segments across each row.

At the top of the next page we see the slope field for $\frac{dy}{dx}=2x-1$.



Note that since the slope was dependent on the value of *x*, we get columns of segments that are parallel.

In the next sketch, we graph the slope field for $\frac{dy}{dx} = -\frac{1}{2}y + 3$.



Note in this case, the slope depended only on *y* and the result is rows of parallel segments.

We do have to be able to generate simple slope fields by hand but the whole slope field concept has become a part of a first course in Calculus because of the introduction of calculators and other software which can generate slope fields for much more complicated equations.