Maximum and Minimum Function Values

Introduction

This next section of material deals with applications of the derivative. We will start with a discussion of the various types of maximum and minimum function values and how we find them. Next, we use the derivative to analyze functions—determining where they are increasing or decreasing, where they are concave up or down, where (if anywhere) they have inflection points and so on. Finally we will do some problems involving applications of function extrema—in the form of optimization problems.

Extrema

The word "extrema" is used when we want to talk about both maximum function values and minimum function values. For instance, a question may ask you to find "the extrema" of a function. This means that you need to find all of the different types of maximum and minimum function values for a function—if indeed it has any at all. Now, there are two types of extrema, absolute and relative. If we want to talk about both relative maximums and relative minimums, we use the term "relative extrema". If we want to discuss both absolute maximums and absolute minimums, we use the term "absolute extrema".

First, a few definitions:

- f has an **absolute maximum** $f(c)$ at $x = c$ if $f(c) \geq f(x)\ \forall x$
- f has an **absolute minimum** $f(c)$ at $x = c$ if $f(c) \leq f(x)\ \forall x$

An absolute minimum is a function value below which the function never goes. An absolute maximum value is a function value above which the function never goes.

Consider $f(x) = x^2 + 1$. This is an upward opening parabola with a vertex at (0,1). The smallest function value this function takes on is 1. We say that f has an absolute minimum of 1 at $x = 0$. Note that f does not have an absolute maximum value. If you find a function value, I can always find a larger one.

Consider $g(x) = -(x-2)^2 + 3$. This is a downward opening parabola with a vertex at (2,3). The largest function value we can ever get is 3. We can say that g has an absolute maximum of 3 at $x = 0$. Note that g does not have an absolute minimum.

Consider $h(x) = x^3$. This cubic has no absolute extrema.

So, we need to realize that not all functions have absolute extrema…unless…unless they are continuous functions defined on a <u>closed</u> interval. Now that's an entirely different situation. If a function is continuous on a closed interval it must have both an absolute maximum and an absolute minimum. In fact, this is a theorem…a very famous theorem.

Extreme Value Theorem

If f is continuous on a closed interval $[a,b]$, then f has an absolute maximum $f(c)$ at some $x=c\in[a,b]$ **and** an absolute minimum $f(d)$ at some $x=d\in[a,b]$.

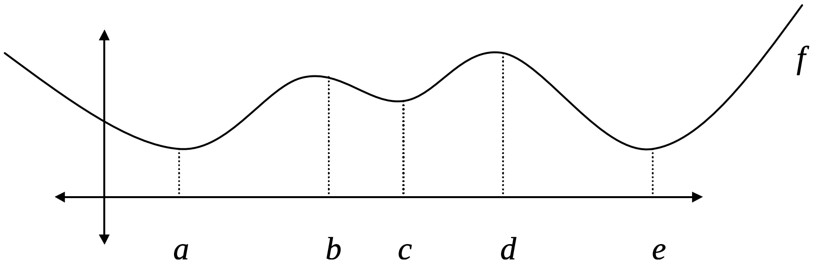
This is an "existence" theorem. It tells us only that these absolute extrema exist, it does not tell us how to find them.

To find these absolute extrema we must first introduce another type of extrema…relative extrema…sometimes called "local" extrema.

Definitions:

- f has a **relative maximum** $f(c)$ at $x=c$ if there exists an open interval I containing c such that $f(c)\geq f(x)\ \forall x\in I$
- f has a **relative minimum** $f(d)$ at $x=d$ if there exists an open interval I containing d such that $f(d)\leq f(x)\ \forall x\in I$

The easiest way to get an initial idea of what a relative extrema is, is to look at the graph below.



The graph of f above has

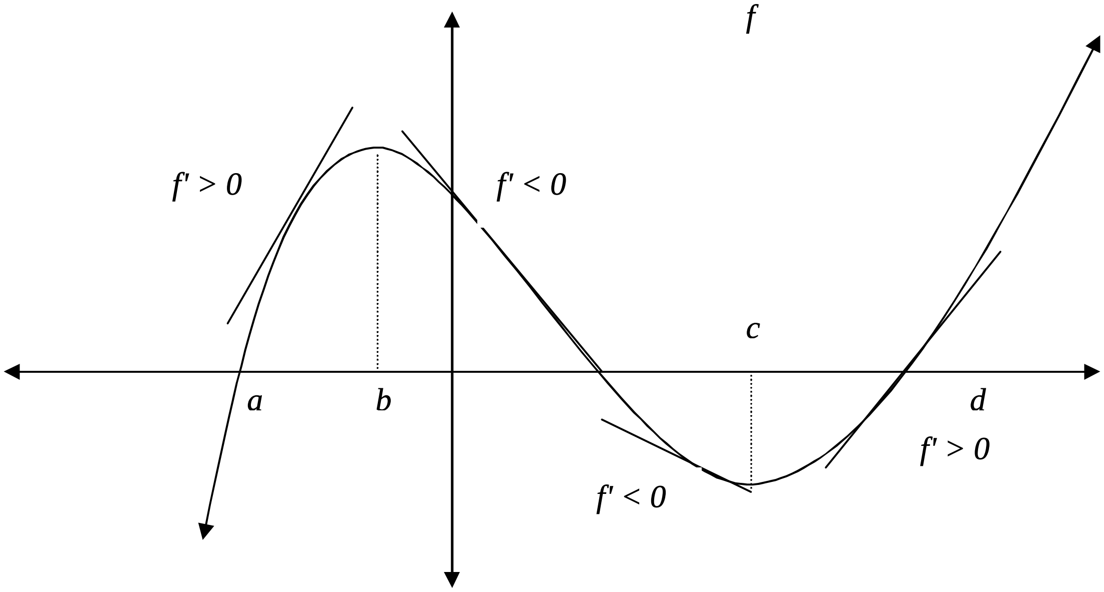
- relative maximums at $x = b$ and $x = d$
- relative minimums at $x = a$, $x =$ c and $x = e$

Now, the function value at $x = a$ or $x = e$ may very well be an absolute minimum. When an extrema is both an absolute and a relative extrema, we would normally call it an absolute extrema…depending on the situation and the question.

We now know what relative extrema are, but haven't yet discussed how to find them. To actually find these relative extrema, we will use our "big hammer"…the derivative.

How did we determine whether or not a function had an inverse? We tried to show that the function was one-to-one. We did this by finding out if the function was always increasing or always decreasing. If the derivative was positive everywhere in an interval, we said that the function was increasing on that interval. If the derivative was negative everywhere on an interval, we said that the function was decreasing on that interval.

Consider the diagram below. It shows a function that increases on (*a*,*b*) and decreases on (*b*,*c*) and increases again on (*c*,*d*).



We can see from the diagram that the derivative is positive before *b* and negative on (*b*,*c*). Since $f'(x) > 0$ on (a,b) and $f'(x) < 0$ on (b,c), then by the Intermediate Value Theorem, $f'(x) = 0$ for some $x \in (a,c)$. In fact, $f'(x) = 0$ at $x = b$. This means that *f* has a relative maximum at $x = b$. The same situation, in reverse, happens at $x = c$, where *f* has a relative minimum.

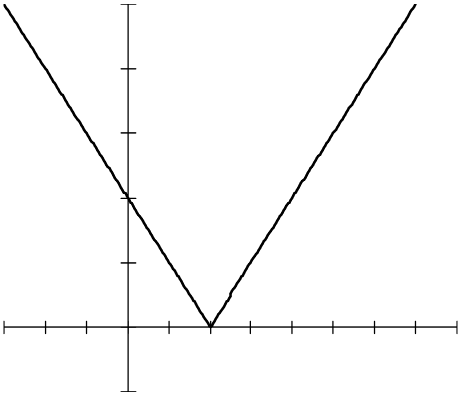
Fermat's Theorem—Part I

If *f* has a relative extrema at $x = c$, then $f'(c) = 0$, if $f'(c)$ exists.

Note that this is an "if, then" theorem, not an "if and only if" theorem. In other words, it doesn't work both ways. Just because $f'(x) = 0$, it does NOT mean that *f* has a relative extrema at $x = c$.

Consider $f(x) = x^3$ and its derivative $f'(x) = 3x^2$. We can see that $f'(x) = 0$ at $x = 0$ but we also know that *f* does not reach a relative maximum or minimum at $x = 0$. Instead, since $f'(x) > 0$ before $x = 0$ and after $x = 0$, *f* is always increasing. Here is a case where the derivative of a function is zero at a number but it has no relative extrema.

Let's take a look at another situation where a function may have a relative extrema. Consider $f(x) = |x-2|$. This is a piecewise function and its derivative would be $f'(x) = \begin{cases} 1 & if \;\; x > 2 \\ -1 & if \;\; x < 2 \end{cases}$. Clearly, when $x > 2$, $f'(x) > 0$ and for $x < 2$, $f'(x) < 0$. We can also see that $f_+'(0) \neq f_-'(0)$, so the derivative does not exist at $x = 2$. f has a relative minimum at $x = 2$, but the derivative does not exist there! Take a look at the graph of f.



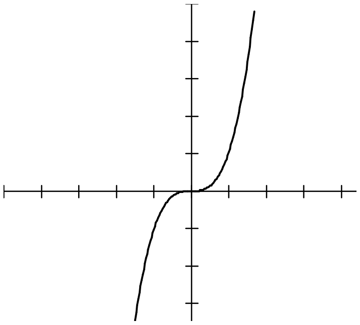
There appears to be two instances where a function can have a relative extrema, where $f'(x) = 0$ or $f'(x) \nexists$. In fact this is a theorem…

Fermat's Theorem

If f has a relative extrema at $x = c$, then the extrema must occur where $f'(c) = 0$ or $f'(c) \nexists$.

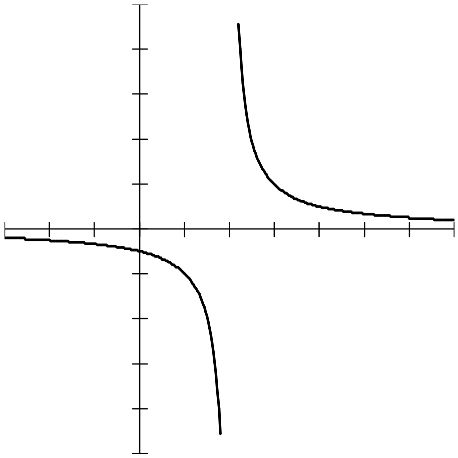
Definition: A **critical number** of a function f is a number c <u>in the domain of f</u> such that either $f'(c) = 0$ or $f'(c) \nexists$.

Fermat's theorem states that if a function has relative extrema, they must occur at critical numbers. This theorem does NOT work in reverse. Just because a function has a critical number does NOT mean it has a relative extrema there. We saw this with $f(x) = x^3$. Its derivative was equal to zero at $x = 0$ but look at the graph…



There is NO relative extrema at $x = 0$! There is a horizontal tangent but no extrema.

Let's take a look at one more function, $f(x)=\frac{1}{x-2}$. Its derivative $f\,'(x)=-\frac{1}{(x-2)^2}$ does not exist at $x=2$. Look at the graph of *f*.



Even though *f '* does not exist at *x* = 2, there is no relative extrema at *x* = 2. There is a vertical asymptote, but no extrema. (Note that *x* = 2 is not in the domain of *f*, so it isn't even a critical number by definition!)

In summary, critical numbers are *x* values (in the domain of the function!) where the derivative is zero or nonexistent.

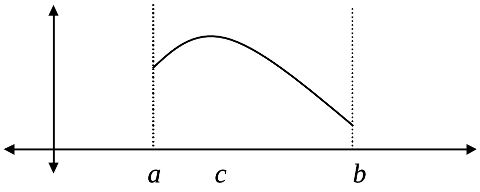
- If a function has relative extrema, they must occur at critical numbers.
- Just because a function has critical numbers does not mean it has relative extrema.

The importance of the two bulleted points above cannot be overstated. Know and understand them!

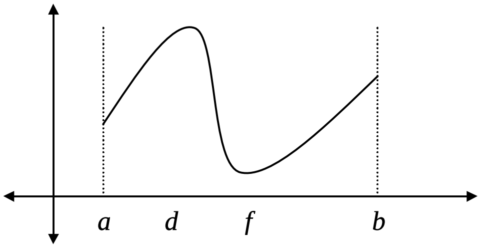
Back to absolute extrema

We started this discussion by talking about absolute extrema. The Extreme Value Theorem tells us that if we have a continuous function on a closed interval, it must have both an absolute maximum and an absolute minimum. The theorem did not tell us how to find them. Using the function values at the endpoints and critical numbers, we can now actually find these absolute extrema.

The absolute extrema of a continuous function on a closed interval must occur at the endpoints or at some critical number between the endpoints.

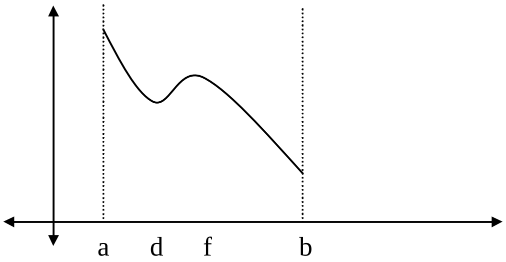


Above is the graph of a continuous function on the closed interval [*a*,*b*]. The absolute maximum will occur at *x* = *c* (a critical number) and the absolute minimum function value will occur at *x* = *b*.



Above is the graph of a continuous function on the closed interval [*a,b*]. Both the absolute maximum and absolute minimum occur at critical numbers. The absolute maximum occurs at $x = d$ and the absolute minimum occurs at $x = f$.

One final example,



In this situation both the absolute maximum and absolute minimum occur at the endpoints of the interval. The absolute maximum occurs at x = a and the absolute minimum occurs at x = b. The extrema inside the interval are only relative extrema—and they do, of course, occur at critical numbers.

In the end, we can have a variety of situations where the absolute extrema occur at some combination of critical numbers or endpoints.

Let's summarize this into a procedure to find the absolute extrema of a continuous function on a closed interval:

- find the critical numbers
- find the function values at the critical numbers
- find the function values at the endpoints of the interval
- the largest of the values found in the previous two steps is your absolute maximum and the smallest function value is the absolute minimum.

Example 1

Find the absolute extrema (if any) of $f(x) = 4x - 1$ on $(-\infty, 8]$.

Notice first that this function is not defined on a closed interval. The graph is a line with a positive slope which extends infinitely in the negative direction but stops at $f(8)$. Since $f(8) = 31$, 31 is the absolute maximum function value. f has no absolute minimum.

Example 2

Find the absolute extrema (if any) of $f(x) = 4x - 1$ on $(-5, 8)$.

The function is not defined on a closed interval and there are no critical numbers since $f'(x) = 4$. Let's look for an absolute maximum first. Notice that as we get closer and closer to 8, the function values continue to get larger and larger. We can continue this process forever…continually getting larger and larger function values (all less than 31). This means that *f* has no absolute maximum! The same logic can be applied to the left end of the interval, resulting in the conclusion that *f* has no absolute minimum either.

Example 3

Find the absolute extrema (if any) of $f(x) = \dfrac{x}{x+1}$ on [1,2].

First find the critical numbers.

$$f'(x) = \frac{1}{(x+1)^2}$$
$$f'(x) \text{ is never zero since } 1 \neq 0$$
$$f'(x) \not\exists \text{ when } x = -1$$

Now, you may be tempted to say that $x = -1$ is a critical number, but it isn't. Remember that in order for a number to be called a <u>critical number</u>, it must be in the domain of the function—which it isn't in this case. Therefore, we have no critical numbers. And besides, it's outside of the interval we are dealing with!

Now find function values at the endpoints…

$$f(1) = \frac{1}{2} \text{ and } f(2) = \frac{2}{3}.$$

$\therefore$ the absolute maximum of *f* on [1,2] is $\dfrac{2}{3}$ and the absolute minimum is $\dfrac{1}{2}$.

Example 4

Find the absolute extrema (if any) of $f(x) = 4x^3 - 15x^2 + 12x$ on [0,3].

This is a continuous function on a closed interval so first find critical numbers.

$$f'(x) = 12x^2 - 30x + 12$$
$$f' \ \exists \ \forall x$$
$$f'(x) = 0 \text{ when } x = \frac{1}{2} \text{ or } x = 2$$

Now find function values at the critical numbers and the endpoints…

$$f(0) = 0$$

$$f(3) = 9$$

$$f(2) = -4$$

$$f\left(\frac{1}{2}\right) = \frac{11}{4}$$

$\therefore$ The absolute maximum is 9 and the absolute minimum is –4.

The Mean Value Theorem

Introduction

The Mean Value Theorem, like the Intermediate Value Theorem belongs to a class of theorems called "existence theorems". Existence theorems guarantee us the existence of some number. The Mean Value Theorem is primarily used in proofs of other theorems. It will tell us something very important about the relationship between the average rate of change in a function and the instantaneous rate of change. Actually we will first discuss a corollary of the Mean Value Theorem, Rolle's Theorem.

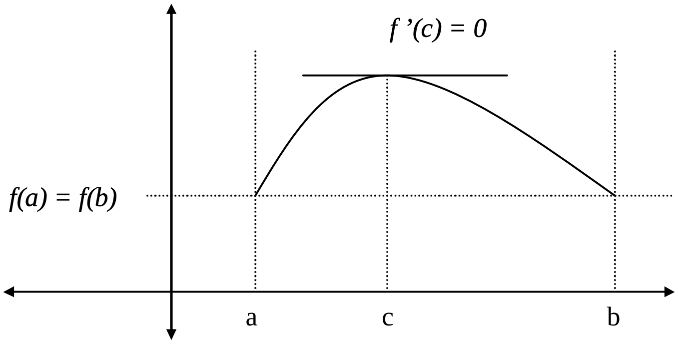
Rolle's Theorem

If f is continuous on $[a,b]$ and differentiable on (a,b) and if $f(a)=f(b)$, then $\exists$ a $c \in (a,b)$ *such that* $f'(c)=0$.

Why does the theorem state that f must be continuous on $[a,b]$ and not just (a,b)? This is because we need to be sure that we can meet the "if" part of the theorem which, in part, says $f(a)=f(b)$. This could not be true unless $f(a)$ and $f(b)$ existed.

Why does the theorem only require that f be differentiable on the open interval (a,b) and not the closed interval $[a,b]$? We need the function to be differentiable because we want to make sure the function is "well behaved" for numbers between a and b. We really cannot insist that the function be differentiable at the endpoints—a function cannot be differentiable at an endpoint because we are always missing either the left or right handed derivative.

This statement essentially tells us that if a function is well behaved on some interval and the function values at the endpoints are equal, then the function must reach a relative maximum or relative minimum at least once between the endpoints. The graph below illustrates the case where the function meets the criteria of the theorem and has one relative extrema.



Most of the problems involving Rolle's Theorem are simple verifications of the theorem.

Example 1

Verify that Rolle's Theorem holds for $f(x)=x^3+x^2-2x+1$ on $[-2,0]$ and then find the c that satisfies the conclusion of the theorem.

Since *f* is a polynomial, *f* is continuous on [–2,0] and differentiable on (–2,0). Also, $f(-2)=1$ and $f(0)=1$. Since all the criteria have been met, Rolle's Theorem holds for *f* on the given interval.

$$f'(x)=3x^2+2x-2$$

Now, $f'(x)=0$ when $x=\frac{-1+\sqrt{7}}{3}$ *or* $x=\frac{-1-\sqrt{7}}{3}$.

Since $\frac{-1+\sqrt{7}}{3}\in(-2,0)$, $c=\frac{-1+\sqrt{7}}{3}$

Of course, questions about Rolle's Theorem can come in many forms. You will need to recognize the theorem in a question. A typical AP question might ask something like:

"If *f* is a function that is continuous on [3,6] and differentiable on (3,6) and f(3) = f(6), then which of the following must be true?" In the list of possible answers, you would be looking for one that states that the value of the derivative of the function at some number between 3 and 6 is equal to zero. Something like "*f '* (some number between 3 and 6) = 0".

Historical note: Rolle's Theorem, named for the French mathematician Michel Rolle, first appeared in 1691. The interesting irony is that after discovering this theorem, Rolle became a fierce critic of the calculus. (Likely because of the problems surrounding "suppressing the h" and other such shady tactics that were used when the calculus was first invented). In fact, he once declared that the calculus was a "…collection of ingenious fallacies." It is truly ironic that despite his vehement opposition to calculus, his name and theorem appear in nearly every calculus text ever written. If he's out there now looking on, perhaps he's happier that our techniques and tactics are now grounded in solid mathematics!

The Mean Value Theorem (for derivatives)

The Mean Value Theorem (for derivatives)

If *f* is continuous on [*a*,*b*] and differentiable on (*a*,*b*), then there exists a $c\in(a,b)$ such that $f'(c)=\frac{f(b)-f(a)}{b-a}$.

The Mean Value Theorem tells us that if a function *f* is continuous on some closed interval and differentiable (remember…all "differentiable" means is that we can find a derivative at any particular number) on the open interval, that for some *x*-value strictly in between *a* and *b*, the instantaneous rate of change in function values (the left side) will be exactly equal to the average rate of change over the interval (the right side).

Again, this is an existence theorem and is primarily used in the proof of other theorems. It guarantees that certain numbers will exist under certain conditions. It does, however, have some interesting applications.

Consider a turnpike where you pass a toll booth at the start and at the end. At the first toll booth you are given a time-stamped receipt. At the end you present the receipt and pay your toll. Well, the people who run the turnpike know exactly how long the road is between toll booths. They also know the speed limit. Knowing these two things, they know exactly how long it should take you to drive from booth to booth. If you finish in less than a specified amount of time, you will automatically be given a speeding ticket at the second toll booth! They do not have to have seen you actually exceed the speed limit. After all, the Mean Value theorem states that the instantaneous rate of change (instantaneous velocity) must be equal to the average rate of change at some point over a specified interval. If you finished too soon, your instantaneous velocity at some point must be equal to your average velocity over the interval. If you were speeding, your average velocity is greater than the specified average…thus at some point you must have exceeded the specified speed limit.

Let’s say the road is 100 miles long and the speed limit is 50 mph. It should take you two hours or more to drive the distance. If you finish in 1 hour 55 minutes, your average velocity will be $\frac{100\,\text{mi}}{115\,\text{min}} = \frac{100\,\text{mi}}{1.917\,\text{hr}} = 52.174\,\text{mph}$. If your average over the interval was 52.174 mph, then by the Mean Value Theorem your instantaneous velocity at some point during the trip was 52.174 mph. Busted!

Example 2

Verify that the Mean Value Theorem holds for $f(x) = 2x^3 + x^2 - x - 1 \ on\ [0,2]$. Then find the value *c* which satisfies the conclusion of the theorem.

To verify that the theorem holds…

- *f* is continuous on [0,2] because polynomial functions are continuous everywhere.
- *f* is differentiable on (0,2) because *f*' is a polynomial function. (Note: $f'(x) = 6x^2 + 2x - 1$ is a polynomial function.)

Now, we need $f'(c) = \frac{f(b) - f(a)}{b - a}$ so

$$6c^2 + 2c - 1 = \frac{17 - (-1)}{2 - 0}$$
$$6c^2 + 2c - 10 = 0$$

$$c=\frac{-1+\sqrt{61}}{6}\ \ or\ \ c=\frac{-1-\sqrt{61}}{6}$$

Since c must be in (0,2),

$$c=\frac{-1+\sqrt{61}}{6}.$$

Like Rolle's Theorem, the questions are not always straightforward. You will have to recognize the Mean Value Theorem in a question. A question may look like…

"If g is continuous on [2,7] and differentiable on (2,7), which of the following must be true?" The answer you are looking for could say something like "$5g'(c)=g(7)-g(2)$ for some c in (2,7)" (Note that if you divide both sides by 5 you get a statement of the Mean Value Theorem.)

Questions like the one above and the similar question involving Rolle's Theorem can either be very, very easy points for you on the AP test or not…depending on whether or not you know the theorems!

Historical note: The Mean Value Theorem was first formulated by Joseph-Louis Lagrange. Lagrange was one of history's greatest mathematicians having contributed to calculus, the theory of functions, celestial mechanics and number theory to name a few. Lagrange was born in Italy and became a professor of mathematics at the age of 19. He eventually succeeded Euler at the Berlin Academy. After some time at the Berlin Academy, Lagrange accepted an appointment from France's King Louis XVI and actually lived in the Louvre. History tells us that Lagrange, unlike some mathematicians of his day, was a kind, gentle and quiet man who dedicated his entire life to science and mathematics.

Derivatives and the Analysis of a Function

Introduction

This section is composed of several topics:

- the first derivative test
- concavity and inflection points
- the second derivative test
- putting it all together—analysis of a function
- finding the graph of *f* from the graph of *f '*

This material represents several sections in a typical calculus textbook. I have lumped them all together because the concepts are so closely related. What we are about to do is use the first and second derivative to analyze a function. By "analyze" I mean we will be able to determine where a function is increasing or decreasing, where it has relative extrema and what type of extrema they are, where a function is concave up or down and whether or not the curve has any inflection points. Now, I know I've just used several terms you are not familiar with…you will be soon enough! In addition, we will consider a third type of asymptote—the oblique asymptote.

The First Derivative Test

The First Derivative Test is a test for relative extrema. We already know that relative extrema occur at critical numbers—the first derivative test will allow us to determine if the relative extrema is a relative maximum or relative minimum. Let's review a couple of definitions first.

f is said to be **increasing** on the interval (a,b) iff $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ where x_1 and x_2 are both in *I*.

f is said to be **decreasing** on the interval (a,b) iff $f(x_1) > f(x_2)$ whenever $x_1 < x_2$ where x_1 and x_2 are both in *I*.

f has a **relative minimum** $f(c)$ at $x = c$ if there exists on open interval *I* containing *c* such that $f(c) \le f(x) \forall x \in I$.

f has a **relative maximum** $f(c)$ at $x = c$ if there exists on open interval *I* containing *c* such that $f(c) \ge f(x) \forall x \in I$.

The First Derivative Test for Relative Extrema

If c is a critical number of f:

If $f'(c) < 0 \ \forall x$ in some open interval having *c* as its right endpoint and $f'(c) > 0 \ \forall x$ in some open interval having *c* as its left endpoint, then *f* has a relative minimum of $f(c)$ at $x = c$.

If $f'(c) > 0 \ \forall x$ in some open interval having c as its right endpoint and $f'(c) < 0 \ \forall x$ in some open interval having *c* as its left endpoint, then *f* has a relative maximum of $f(c)$ at $x = c$.

Example 1

Find the relative extrema (if any) of $f(x) = x^3 - 9x^2 + 15x - 5$.

$$f'(x) = 3x^2 - 18x + 15$$
$$f' \exists \; \forall x$$
$$f'(x) = 0 \text{ when } 3(x-1)(x-5) = 0$$
$$x = 1 \text{ or } x = 5$$

x	$f(x)$	$f'(x)$
$x<1$		+
$x=1$	2	0
$1<x<5$		–
$x=5$	–30	0
$x>5$		+

From the chart above,
Since $f'(x) > 0$ on $(-\infty, 1)$ and $f'(x) < 0$ on $(1,5)$, f has a relative maximum of 2 at $x = 1$.
Since $f'(x) < 0$ on $(1,5)$ and $f'(x) > 0$ on $(5,\infty)$, f has a relative minimum of –30 at $x = 5$.

Example 2

Find the relative extrema (if any) of $f(x) = \dfrac{x-2}{x+2}$

$$f'(x) = \frac{4}{(x+2)^2}$$

$f' \nexists$ at $x = -2$ (Note this is NOT a critical number—it is not in the domain of f but we will however, use it in our chart.)

$f'(x)$ never equal to zero.

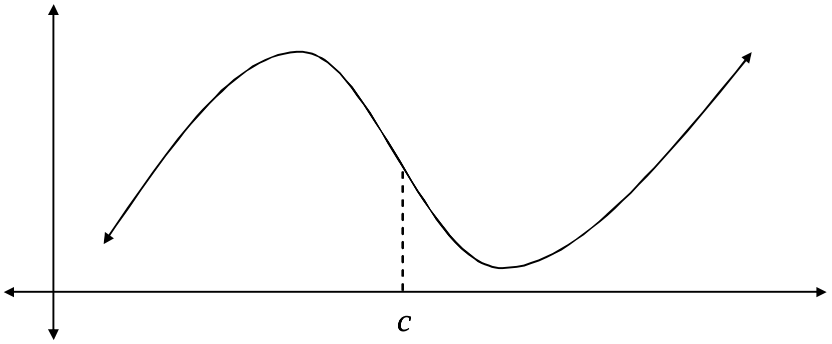
x	$f(x)$	$f'(x)$
$(-\infty, -2)$		+
$x = -2$	$\nexists$	$\nexists$
$(-2, \infty)$		+

From the chart above we can see that f is always increasing and has no relative extrema.

Note: Example 2 brings up an important point. There is a difference between just being asked to find critical numbers and being asked to find relative extrema. Even though $x = -2$ was not a critical number, we still use it in a chart to determine where f is increasing or decreasing. When analyzing a function, include all values of x where the derivative is zero or non-existent…even if they are not "critical numbers". We include them in charts because they are values where f' could possibly change signs. If asked to find critical numbers, be sure to avoid calling a value of x a critical number if it is not in the domain of f.

Concavity and Points of Inflection

Consider the following sketch of a function f.



Notice that on $(-\infty, c)$, any tangent that is drawn will be above the curve. Notice also that on $(-\infty, c)$ the value of $f'(x)$ goes from being some positive number, to zero, to some negative number. In other words, $f'(x)$ is getting smaller. This means that the rate of change in $f'(x)$ is negative. The rate of change in $f'(x)$ is $f''(x)$. On the interval $(-\infty, c)$, $f''(x) < 0$.

Definition: If $f''(x) < 0$ on (a,b), we say that the curve is **concave down** on the interval.

Note also that if a tangent to a curve is above the curve, the curve at that point is concave down! This is a quick visual check for concavity.

Let's do the same analysis on the interval (c, ∞). Notice that on (c, ∞), any tangent that is drawn will be below the curve. Notice also that on (c, ∞) the value of $f'(x)$ goes from being some negative number, to zero, to some positive number. In other words, $f'(x)$ is getting larger. This means that the rate of change in $f'(x)$ is positive. The rate of change in $f'(x)$ is $f''(x)$. On the interval (c, ∞), $f''(x) > 0$.

Definition: If $f''(x) > 0$ on (a,b), we say that the curve is **concave up** on the interval.

Again notice that any tangent drawn on the interval (c, ∞) will be below the curve…a nice, quick visual way to check concavity.

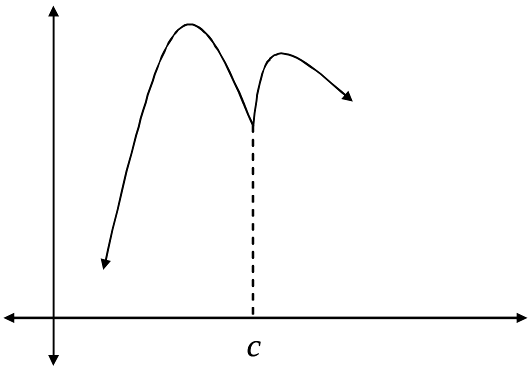
Back to our sketch. Before c, we determined that $f''(x) < 0$ and after c we determined that $f''(x) > 0$. Well, by the Intermediate Value Theorem, there must be a point inbetween where $f''(x) = 0$! At $x = c$, $f''(x) = 0$ and it was the point where the concavity of the curve changed.

Definition: A point where a curve changes concavity is called an **inflection point**.

When we try to find inflection points, we look for values of x where $f''(c) = 0$ *or* $f''(c) \nexists$. These values of x are not called critical numbers! The term "critical numbers" is reserved for the first derivative. Values of *x* where the second derivative is zero or non-existent are called "possible inflection points".

We know that relative extrema must occur at critical numbers but the existence of a critical number does not necessarily mean a function has a relative extrema. We make a chart to verify if a function has a relative extrema at any particular critical number. The same goes for possible inflection points. Once we find possible inflection points, we must make a chart to determine if there actually is a change in concavity.

Consider the sketch of a function *f* below.



Notice that $f''(x) \nexists$ at $x = c$ but the curve is concave down on both sides of *c*. There is no inflection point at $x = c$.

Example 3

Determine where the graph of $f(x) = x^3 - 3x + 1$ is concave up and concave down. Find any inflection points.

$$f'(x) = 3x^2 - 3$$
$$f''(x) = 6x$$

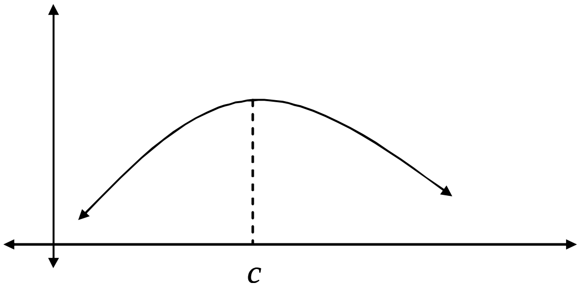
- $f'' \exists \; \forall x$
- $f''(x) = 0$ when $x = 0$

x	$f(x)$	$f''(x)$
$(-\infty, 0)$		$-$
$x = 0$	1	0
$(0, \infty)$		$+$

From the chart above, *f* is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$. *f* has an inflection point at (0,1).

The Second Derivative Test for Relative Extrema

First off…the Second Derivative Test is NOT a test for concavity. It is a test for relative extrema—and it's clever and fast for many problems. Let's say you were faced with a situation in which you had a critical number at $x = c$ and at the same time you know the curve is concave down in the delta-deleted neighborhood of c. ("Delta-deleted neighborhood of …" is an expression we use to denote a region just to the left and right of a number. It is a way to talk about an interval without actually specifying an interval—it means "close to c".) The only possiblility is that there is a relative maximum located at $x = c$! Take a look at the relative maximum below.



A similar situation would arise if $f'(c) = 0$ and the curve is concave up in the delta-deleted neighborhood of c. In this case the function would have to have a relative minimum at $x = c$.

The Second Derivative Test for Relative Extrema

If $f'(c) = 0$ and $f''(c) > 0$ then f has a relative minimum at $x = c$.

If $f'(c) = 0$ and $f''(c) < 0$ then f has a relative maximum at $x = c$.

Note that the test will not work if $f''(c) = 0$ *or if* $f''(c) \nexists$.

The reason the test will not work if $f''(c) = 0$ or if $f''(c) \nexists$ is that we need to know if the curve is concave up or down—the second derivative must be positive or negative, not zero or non-existent.

If you try to use this test and find out that $f''(c) = 0$ or if $f''(c) \nexists$, you have to go back to the First Derivative Test for Relative Extema…which means you basically have to make a chart with the first derivative to determine relative extrema.

The second derivative test works especially well with polynomial functions. If you've got a complicated function your best tactic may be to stay with the first derivative test.

Example 4

Use the Second Derivative Test to find the relative extrema of $f(x) = x^3 - 9x^2 + 15x - 5$.

$$f'(x) = 3x^2 - 18x + 15$$
$$f' \exists \forall x$$
$$f'(x) = 0 \text{ when } 3(x-5)(x-1) = 0$$
$$x = 5 \text{ or } x = 1$$

$$f''(x) = 6x - 18$$

Since $f''(5) = 12$, *f* is concave up at x = 5 $\therefore$ by the second derivative test *f* has a relative minimum of $f(5) = -30$ at x = 5.

Since $f''(1) = -12$, *f* is concave down at x = 1 $\therefore$ by the second derivative test *f* has a relative maximum of $f(1) = 2$ at x = 1.

Curve Sketching (Analysis of a Curve)

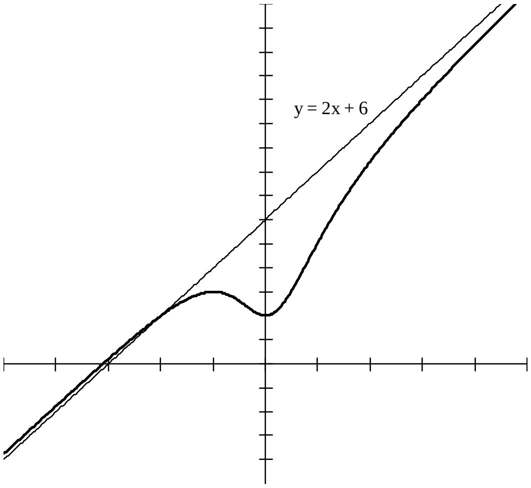
In this section we simply put together all the tools we have learned to analyze a function. In the days before graphing calculators and computers, the primary technique we used to get an accurate sketch of a graph involved analyzing a function with the first and second derivative. Since we now have graphing calculators and computers, we do not necessarily need to use the derivatives to get a sketch of a function. The focus changed from using the derivative to get an accurate sketch to using the derivative to explain why a curve looks like it does. It is also important to be able to do the analysis without the aid of electronic devices—at least for relatively simple functions.

The process of analyzing a function includes the following:

- Finding the domain
- Finding zeros
- Finding any asymptotes
- Using the first derivative to determine where the function increases and decreases
- Using the first or second derivative test to determine relative extrema
- Using the second derivative to determine concavity and inflection points

The only additional piece of information you need has to do with something called an **oblique or slant asymptote**.

Consider the graph of $f(x) = \dfrac{2x^3 + 6x^2 + 2}{x^2 + 1}$.

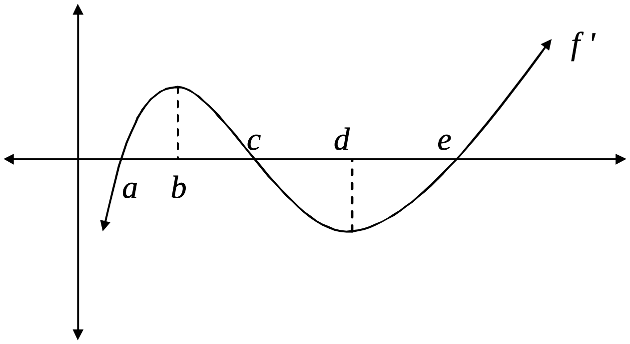


Notice that as $x \to \pm\infty$, the curve approaches the line $y = 2x + 6$. This line is called a oblique asymptote. Oblique asymptotes occur when we have a rational function in which the degree of the numerator is one greater than the degree of the denominator. To find the equation of the asymptote, simply use polynomial long division and "ignore" the remainder. We don't actually ignore the remainder, but as $x \to \pm\infty$, the remainder goes to zero.

You now have all the tools you need to do a complete analysis of a function--and graph it by hand if you need. You can now explain why the graph of a function looks like it does.

Graphs of *f* and *f* '

One of the classic problems in calculus involves sketching the graph of a function given only the graph of the derivative of the function. Consider the following sketch of the derivative of a function *f*.



Now,we can see that *f* ' is clearly below the *x*-axis on $(-\infty, a)$ and (c, e). This means that $f'(x) < 0$ on $(-\infty, a) \cup (c, e)$ which in turn means that *f* is decreasing on these intervals.

We can also see that $f\,'$ is above the x-axis on (a,c) *and* (e,∞). This means that $f\,'(x) > 0$ on $(a,c) \cup (e,\infty)$ which means that f is increasing on these intervals.

At $x = a$ and $x = e$, $f\,'(x)$ changes from negative to positive and thus by the first derivative test, f must have a relative minimum at $x = a$ and $x = e$. (At both of the points the function decreases, then increases—reaching a minimum.)

At $x = c$, $f\,'(x)$ changes from positive to negative and thus by the first derivative test, f must have a relative maximum at $x = c$. (The function increased and then decreased—reaching a maximum.)

We already know a great deal about this function. Can we determine concavity? Can we locate inflection points? Of course we can!

Notice that on $(-\infty,b)$ the derivative is increasing. This means that its (the derivative's) rate of change is positive…it means that $f\,''(x) > 0$ on this interval. Clearly, f is concave up on $(-\infty,b)$! f is also concave up on (d,∞). Why? Because $f\,'$ is increasing so $f\,''(x) > 0$.

On (b,d) $f\,'$ is decreasing which means $f\,''(x) < 0$…so f is concave down on (b,d).

Finally, inflection points. Note that at $x = b$, the derivative is increasing just before b and decreasing just after b. This means that $f\,''(x)$ went from positive to negative at $x = b$. There's an inflection point at $x = b$! Isn't this fun? Last but not least, note that at $x = d$, the derivative is decreasing just before d and increasing just after d. This means that $f\,''(x)$ went from negative to positive at $x = d$. There's an inflection point at $x = d$!

So…if someone gives you the graph of the derivative of a function, you can sketch the graph of the function itself. I would like you to be able to use the kind of thinking that we did in the problem above but for those who would rather memorize…here are the facts. (Actually, you'll do this enough that you'll end up being able to graph functions from their derivatives so fast that it will be like you memorized it.)

If $f\,'$ crosses the x-axis from below to above, f has a relative maximum.
If $f\,'$ crosses the x-axis from above to below, f has a relative minimum.
If $f\,'$ is above the x-axis, f is increasing.
If $f\,'$ is below the x-axis, f is decreasing.
If $f\,'$ is increasing, f is concave up.
If $f\,'$ is decreasing, f is concave down.
Relative extrema on $f\,'$ are inflection points on f.

Applied Maximum and Minimum Problems (Optimization Problems)

Introduction

Optimization problems are an application of the Extreme Value Theorem. Remember, this theorem tells us that if a function is continuous on a closed interval, it must have an absolute maximum and an absolute minimum. These absolute extrema must occur at the endpoints or at a critical number. For most problems, even though they are applications of the Extreme Value Theorem, we won't have to find the interval on which the problem is taking place. Most of the solutions we find will occur at critical numbers. If the result you get at a critical number is meaningless in terms of your problem, that's when you'll have to find the endpoints and find function values there. This will be clearer as we move through some examples.

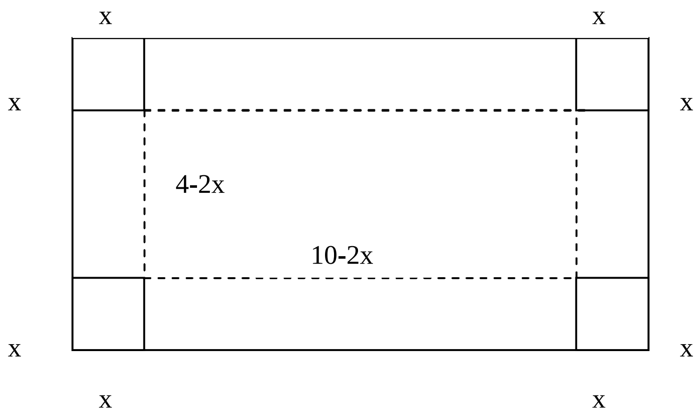
The process is simple:

- Determine what quantity is to be optimized.
- Find a function, in a single variable, which describes this quantity.
- Find the critical numbers for this function.
- Find function values at these critical numbers and choose the answer which makes sense for your problem. (Most of the time you will only find one critical number.)

Example 1

A box with an open top is to be constructed from a sheet of cardboard 4 feet wide and 10 feet long by cutting squares from each of the four corners and folding up the sides. Find the largest volume that such a box can have.

First, start with a diagram.



Now, the volume of the resulting box is given by

$$V(x) = x(4-2x)(10-2x)$$

or

$$V(x) = 4x^3 - 28x^2 + 40x$$

Taking a derivative and finding critical numbers gives us

$$V'(x) = 12x^2 - 56x + 40$$

Critical Numbers

$V' \exists \forall x$

$V'(x) = 0$ when $x = \dfrac{7-\sqrt{19}}{3}$ or $x = \dfrac{7+\sqrt{19}}{3}$

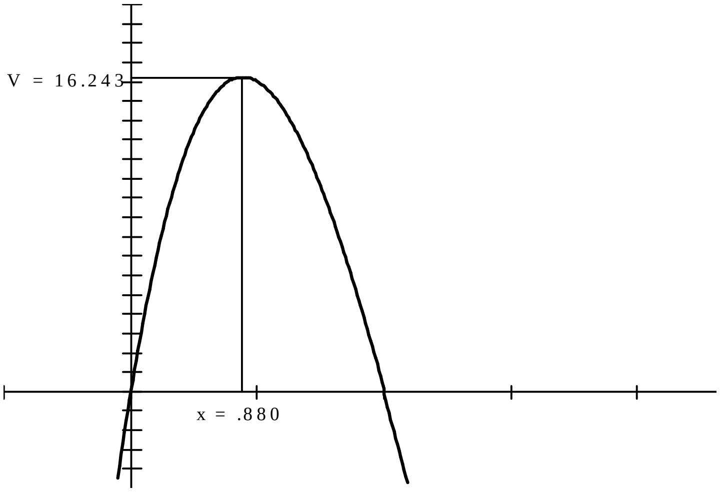
$x \approx .880$ or $x \approx 3.786$

Now, we obviously cannot make a cut of 3.786 feet at each corner…the short side is only 4 feet! So the maximum volume will occur when the square measures .880 feet on a side. Find $V(.880)$.

$$V(.880) = 16.243$$

Therefore, the maximum volume of the box will be 16.243 square feet.

Let's take a look at the problem graphically. Below is the graph of the volume function. Note that the critical number x = .880 yields the maximum function value on the interval [0,2]. We actually used this interval because the smallest cut we could make is zero feet and the largest would have been 2 feet. All we did was find where the curve had a horizontal tangent! That's really all optimization problems are about!



Example 2

Find two numbers whose difference is 100 and whose product is a minimum.

If two numbers differ by 100, we can call one of them *x* and the other *x* + 100.

Their product is given by:

$$P(x) = x(x+100) = x^2 + 100x$$

Now,

$$P'(x) = 2x + 100$$

Critical Numbers

$P' \exists \forall x$

$P'(x) = 0$ when $x = -50$

Therefore, the two numbers are –50 and 50.

This result can be verified graphically. (Which of course you should do!)

Example 3

Find the point on the parabola $x + y^2 = 0$ that is closest to the point (0, –3).

We need to minimize the distance…so we will use the distance formula.

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Here's something to remember when dealing with this type of distance problem. Minimizing the square root of a number is the same a minimizing the number itself. Smaller numbers have smaller square roots. The same goes for maximizing. What this means for us is that we can use the distance formula without the radical and get the same results we would get with the radical included!

We can do the problem in terms of either *x* or *y*. If we solve for y, we will need to consider both the positive and negative $\sqrt{x}$ …which is unnecessarily messy. Instead we solve for *x* and get

$$x = -y^2$$

Any point on the parabola can be written $(-y^2, y)$, so we will use the distance formula with $(-y^2, y)$ and (0, –3) to get a function in a single variable which describes the distance.

So, the function in a single variable we will be using is

$$D(y)=(-y^2-0)^2+(y+3)^2$$

or

$$D(y)=y^4+y^2+6y+9$$

Differentiating,

$$D'(y)=4y^3+2y+6$$

Critical numbers

$D' \exists \forall x$

$D'(y)=0$ when $y=-1$

Substituting $y = -1$ into the original parabola yields $x = -1$.

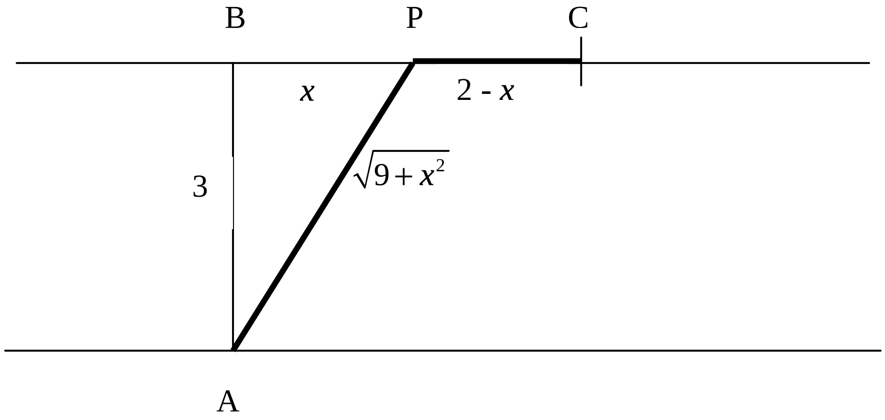
Therefore, the point on $x+y^2=0$ closest to (0,–3) is (–1,–1).

Example 4

Points A and B lie on opposite sides of a river 3 km wide. Point C is on the same side as B but is 2 km downstream. A cable company wants to lay a cable from A to C. The cost of laying the cable under water is $12,500 per kilometer and the cost of laying the cable on land is $10,000 per kilometer. What should the cable company do to minimize the cost of laying the cable from A to C?

First, consider the possibilities. They can lay the cable directly from A to C (entirely under water), they can go directly from A to B and then to C (the least amount under water), or they can go to a point P somewhere between B and C. It's best to work the problem from the last standpoint…otherwise it's just an algebra problem.

First, a diagram.



We let the distance from B to P be *x*, which leaves the distance from P to C as 2 – *x*. The diagonal distance across the river is determined from the Pythagorean Theorem. Now we need a function which describes the cost of laying the cable. Since the cost under water is \$12,500 per kilometer, the cost from our diagram is $12,500\sqrt{9+x^2}$. The cost over land is \$10,000 per kilometer, so the cost from our diagram is $10,000(2-x)$. We can now write a function in a single variable to describe the cost.

$$C(x)=12,500\sqrt{9+x^2}+10,000(2-x)$$

Differentiating yields

$$C'(x)=\frac{12,500x-10,000\sqrt{9+x^2}}{\sqrt{9+x^2}}$$

<u>Critical Numbers</u>

Since $9+x^2>0\ \forall x,\ C'\ \exists\ \forall x$

$C'(x)=0$ when $x=4$

Now, *x* = 4 would normally be the value we are looking for but if the company goes directly from A to C, the most *x* can be is 2. This is an example of a problem whose solution lies at an endpoint. Now find *C*(0) for the cost from A to B then to C and *C*(2) which would be the cost of going directly from A to C.

$$C(0)=57,500$$
$$C(2)=45,069.39$$

Therefore, the minimum cost will be \$45,069.39 by going entirely underwater from A to C.