Inverse Functions

Introduction

Inverses are pairs of functions in which the input and output have been reversed. If *f* is {(1,2),(3,4),(5,6)}, then its inverse would be {(2,1),(4,3),(6,5)}. Note the *x* and *y* values have been "switched"—the domain of the first becomes the range of the second and the range of the first becomes the domain of the second. This fact will finally give us a way to analytically find the range of some functions…something we could not do very often before.

Inverses are essentially pairs of functions that "undo" each other. The statement that the functions "undo" each other can be written

> If f and g are inverses,
> $f(g(x)) = x$ and $g(f(x)) = x$

Example 1

Consider $f(x) = x + 2$ and $g(x) = x - 2$. Note that

$$f(g(x)) = f(x-2) = (x-2)+2 = x$$
and
$$g(f(x)) = g(x+2) = (x+2)-2 = x$$

Because $f(g(x)) = x$ and $g(f(x)) = x$, *f* and *g* are inverses.

This is how we prove that two functions are inverses!

One important difference between our new definition of "inverses" from what you may have learned before is this…for a function to have an inverse, the inverse itself must be a function. Previously, you most likely found many inverses by "switching the *x* and *y*" and solving for *y*. You could do this with most any function. You called the resulting equation the inverse of the original. Now, you can only call the resulting equation an inverse if it is itself a function.

Consider $f(x) = x^2$. Now, using previously learned techniques, we "switch the *x* and *y*" and solve for *y*.

$$y = x^2$$
$$x = y^2$$
$$y^2 = x$$
$$y = \pm\sqrt{x}$$

The resulting equation is NOT a function so we cannot call it the "inverse" of $f(x) = x^2$. In fact, $f(x) = x^2$ has no inverse. We will learn why in a few moments.

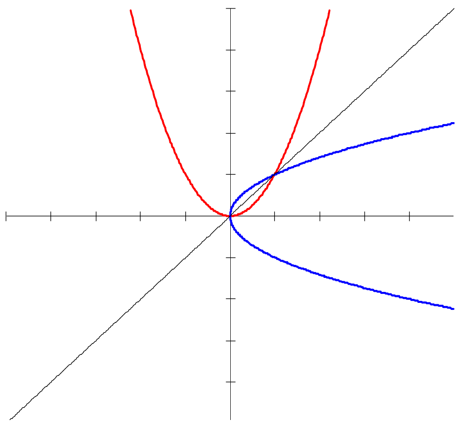
One more bit of notation…we normally denote a function and its inverse by f and f^{-1} so we can also write

$$f(f^{-1}(x)) = x \text{ and } f^{-1}(f(x)) = x$$

Determining if a function has an inverse

When the input and output are reversed, the graph of a function is reflected across the line $y = x$. This being true, if an "inverse" must be a function in order for it to be an inverse, it must pass a vertical line test.

Below is the graph of $f(x) = x^2$ and $g(x) = \pm\sqrt{x}$. g is the result of "switching the x and y" and solving for y. The upward opening parabola is, of course, f. The top half of the sideways opening parabola is $g(x) = \sqrt{x}$ and the bottom half is $g(x) = -\sqrt{x}$.



You can clearly see that g is not a function. Because g is not a function, it cannot be the inverse of f. Notice that g does not pass a vertical line test. Also notice that f would not pass a horizontal line test. This is because a vertical line reflected across the line $y = x$ is a horizontal line! If a function cannot pass a horizontal line test, its "inverse" will not pass a vertical line test…and thus will not be an inverse. So, if a function does not pass a horizontal line test, it does not have an inverse.

So, what type of functions would pass a horizontal line test? They would have to be functions in which each distinct member of the domain was paired with a distinct member of the range. Functions where this is true are called one-to-one, or monotonic. In the example above, the ordered pairs (2, 4) and (–2,4) are both on f. In this case, we have distinct members of the domain paired with the same range value—meaning that f was not one-to-one.

For a function to have an inverse, it must be one-to-one. Another way to state this is
If $x_1 \neq x_2$ *then* $f(x_1) \neq f(x_2)$

Now, there is a more sophisticated way to show whether or not a function is one-to-one. As a matter of fact, a horizontal line test is rarely enough. It would only be a sufficient argument if we were looking at the sketch of a graph (without knowing the actual function) and were asked if the function is one-to-one. What we need is an analytic technique which will determine if a function is one-to-one or not.

Notice on the graph of $f(x)=x^2$, as we move from left to right, the graph "decreases" until zero and then it "increases". Functions that fail a horizontal line test will always decrease and then increase or increase and then decrease—resulting in the same range value for several domain values. For $f(x)=x^2$, the slope of a tangent line drawn for any value of $x<0$ would be negative and the tangent line at any value of $x>0$ would be positive. This should make good sense. If the slope of a tangent line is negative, then rate of change in function values is negative (a negative derivative) and the function values must be getting smaller…the function is decreasing. If the rate of change in function values (the derivative) is positive, the rate of change is positive and function values must be getting larger…the function is increasing.

Functions which are one-to-one are always increasing or always decreasing. This means that a function is one-to-one if the derivative is always positive or always negative!

> If $f'(x)>0\ \forall x$ or if $f'(x)<0\ \forall x$ then f is one-to-one.

This is precisely the analytic technique we were looking for.

So, to show or prove that a function has an inverse, show that it is always increasing or always decreasing by showing that its derivative is always positive or always negative.

Example 2

Determine if $f(x)=x^2$ has an inverse. (Or…Determine if….is one-to-one. Same question.)

This will be a "chart" problem since we want to know where the derivative is positive or negative.

$$f'(x)=2x$$

- $f'\ \exists\ \forall x$
- $f'(x)=0$ when $x=0$

x	$f(x)$
$(-\infty, 0)$	–
$x = 0$	0
$(0, \infty)$	+

Because f' changed signs, f is not always increasing or always decreasing and therefore f is not one-to-one and thus does not have an inverse.

Example 3

Consider $f(x) = \sqrt[3]{x}$. Does *f* have an inverse?

$$f'(x) = \frac{1}{3\sqrt[3]{x^2}}$$

- $f' \nexists$ when $x = 0$
- $f'(x) \neq 0$

x	$f'(x)$
$(-\infty, 0)$	+
$x = 0$	$\nexists$
$(0, \infty)$	+

Because $f'(x) > 0 \ \forall x \neq 0$, f is always increasing, therefore *f* is one-to-one, therefore it has an inverse.

A derivative can pass through a value where it is zero or non-existent and still be considered always increasing or always decreasing.

Finding an inverse of a function

I know that you have always found inverses by "switching the *x* and *y*" and solving for *y*. The only problem with this is that there is no operation in mathematics known as "switch the *x* and *y*". Nonetheless, I will allow you to do that "switching" thing. I will however, show you the proper way to do this. For inverses, the *x* in one function is the *y* in the other and visa versa. In terms of notation we say:

$$y = f^{-1}(x) \ and \ x = f^{-1}(y)$$

This simply means that *x* is the inverse of y and y is the inverse of x.

To find an inverse:

- Let $y = f(x)$
- Solve for x
- Substitute $x = f^{-1}(y)$
- Now you will have a normal function called $f^{-1}(y)$
- Find $f^{-1}(x)$ the same way you would find any other function value

Example 4

If $f(x) = 7x + 5$, find $f^{-1}(x)$.

$$\text{Let } y = f(x)$$
$$y = 7x + 5$$
$$x = \frac{y-5}{7}$$
$$\text{Since } x = f^{-1}(y)$$
$$f^{-1}(y) = \frac{y-5}{7}$$
$$f^{-1}(x) = \frac{x-5}{7}$$

Example 5

If $f(x) = \dfrac{x-2}{x+2}$, find $f^{-1}(x)$.

$$\text{Let } y = f(x)$$
$$y = \frac{x-2}{x+2}$$
$$y(x+2) = x - 2$$
$$yx + 2y = x - 2$$
$$yx - x = -2y - 2$$
$$x(y-1) = -2y - 2$$
$$x = \frac{-2y-2}{y-1}$$
$$\text{Since } x = f^{-1}(y)$$
$$f^{-1}(y) = \frac{-2y-2}{y-1}$$
$$f^{-1}(x) = \frac{-2x-2}{x-1}$$

Domain and Range with Inverses

When we find an inverse, the ordered pairs are reversed. This leads us to conclude:

The domain of f is the range of f^{-1}.
The range of f is the domain of f^{-1}

The domain of f^{-1} is the range of f
The range of f^{-1} is the domain of f

One very important aspect of this relationship is that we finally have an analytic method to find the range of a function—as long as the function has an inverse.

To find the range of a function, find the domain of its inverse—after all, domains are easy to find. The domain of the inverse is the range of our function!

Graphing an inverse

To graph an inverse, reflect the graph across the line y = x. That's it!

Can we find all inverses?

The Fundamental Theorem of Algebra basically tells us that an n^{th} degree equation has *n* solutions. It does not say we can ever find them, it just states how many solutions there will be. The fact of the matter is that there exist equations which we just cannot solve. It has been proven that not all 5^{th} degree equations or higher are solvable.

Consider $f(x) = x^5 + 5x^3 + 2x - 4$. Let's see if f has an inverse.

$$f'(x) = 5x^4 + 15x^2 + 2$$

Now, each of the three terms is greater than zero for all x, so $f'(x) > 0 \forall x$. This means that f is always increasing and thus one-to-one…which means it has an inverse.

Now let's try to find it! Go ahead and "switch the *x* and *y*" if you want…

$$x = y^5 + 5y^3 + 2y - 4$$

When we get to the point where we solve for *y*, we get stuck. Our equation is one of those higher ordered equations which is not solvable! The Fundamental Theorem of Algebra will guarantee us *n* solutions for an n^{th} degree equation, but does not guarantee us that we can find it.

We are faced with a situation in which we know a function has an inverse but we can never find it. Does this mean we cannot know anything about its inverse? Of course not. If we couldn't, our lesson would be over now…but it ain't! In fact, we can determine the slope of a tangent to an inverse without ever even being able to find the inverse itself—much less the derivative of the inverse. (It's kind of tough to find the derivative of a function you can't find in the first place!) Ahh…the mysteries of mathematics!

The Derivative of an Inverse

We actually face several scenarios when we try to find the derivative of an inverse. First of all, if we can actually find the inverse, we can, of course, find its derivative. If all we want to know is the value of the derivative of the inverse at a specific value however, we do not need to find the inverse itself.

Example 6

Let us consider $f(x) = x^3$. Find (a) the inverse, (b) the derivative of the inverse and (c) the value of the derivative of the inverse at $x = 8$.

(a)

$$\text{Let } y = f(x)$$
$$y = x^3$$
$$x = \sqrt[3]{y}$$
$$\text{Since } x = f^{-1}(y)$$
$$f^{-1}(y) = \sqrt[3]{y}$$
$$\therefore\ f^{-1}(x) = \sqrt[3]{x}$$

(b)

$$(f^{-1})'(x) = \frac{1}{3\sqrt[3]{x^2}}$$

(c)

$$(f^{-1})'(8) = \frac{1}{12}$$

We now know the slope of a tangent to the inverse at $x = 8$.

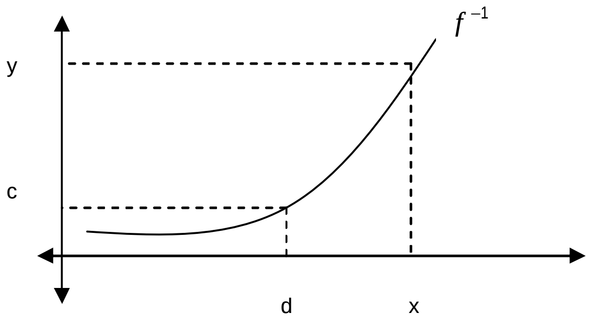
Example 7 shows us the most straightforward problem we can face. We know the function has an inverse, the inverse is easy to find, the derivative of the inverse is easy to find. All of this, however, is not always true.

Is there a way for us to find the value of the derivative of an inverse without actually finding the inverse?

Notice that in Example 7, the (2,8) is on f and that $\frac{1}{f'(2)} = \frac{1}{12}$. This is the same value as $(f^{-1})'(8)$.
This rather surprising result is one of the most important theorems we learn this year.

If (c,d) is on f (which means (d,c) is on f^{-1}),
then $(f^{-1})'(d) = \frac{1}{f'(c)}$

Proof



Consider two points on the graph of f^{-1}, (d,c) and (x,y).
Already we can say several things which we will use in the proof:

- Since (d,c) is on f^{-1} we know that (c,d) is on f
  - This means that $f^{-1}(d) = c$ and $f(c) = d$
- Since (x,y) is on f^{-1} we know that (y,x) is on f
  - This means that $f^{-1}(x) = y$ and $f(y) = x$

$$\text{Now, } (f^{-1})'(d) = \lim_{x \to d} \frac{f^{-1}(x) - f^{-1}(d)}{x - d}$$

This is just the definition of the derivative of f^{-1} at d.

We know $f^{-1}(x) = y$ and $f^{-1}(d) = c$. Substituting we get,

$$(f^{-1})'(d) = \lim_{x \to d} \frac{y - c}{x - d}$$

We also know $f(y) = x$ and $f(c) = d$. Substituting again we get,

$$(f^{-1})'(d) = \lim_{x \to d} \frac{y - c}{f(y) - f(c)}$$

From the graph, we can see that as $x \to d, y \to c$ so,

$$(f^{-1})'(d) = \lim_{y \to c} \frac{y - c}{f(y) - f(c)}$$

Dividing both numerator and denominator by $y-c$ yields

$$(f^{-1})'(d)=\lim_{y\to c}\frac{\frac{y-c}{y-c}}{\frac{f(y)-f(c)}{y-c}}$$

The limit of the numerator is 1 and the limit of the denominator is, by definition of derivative, $f'(c)$!

Thus,

$$(f^{-1})'(d)=\frac{1}{f'(c)}$$

We can use this theorem to find the value of the derivative of an inverse without ever finding the inverse itself!

Example 7

If $f(x)=x^5-x^3+2x$, find $(f^{-1})'(2)$

We do not need to find f^{-1} or $(f^{-1})'(x)$.

Start by finding $f'(x)$.

$f'(x)=5x^4-3x^2+2$

Now, since we want to find $(f^{-1})'(2)$ using our new theorem, the *d* = 2.

The point (*c*, 2) is on *f*…so we need to find *c*

$$\begin{gathered}2=x^5-x^3+2x\\ x=1\\ \therefore c=1\end{gathered}$$

Now apply our theorem,

$$\begin{aligned}(f^{-1})'(2)&=\frac{1}{f'(1)}\\ &=\frac{1}{4}\end{aligned}$$

We have determined that if a tangent is drawn to the inverse of f at x = 2, its slope will be $\frac{1}{4}$ …and we did it without ever finding the inverse!

Exponential Functions and their Derivatives

Introduction

Exponential functions are functions of the form $f(x) = a^x$ where a is a positive constant. Some examples would be $f(x) = 7^x$, $f(x) = 2^x$ and most importantly for us right now, $f(x) = e^x$. Power functions have a variable raised to a numeric exponent but exponential functions involve a positive constant raised to a variable.

One of the many differences between power functions and exponential functions is in their rates of growth…how quickly their values increase. Exponential functions will increase at a tremendously higher rate than a power function.

Consider $g(x) = x^2$ and $f(x) = 2^x$. Let's make a chart of some selected function values.

x	$g(x)$	$f(x)$
0	0	1
1	1	2
2	4	16
5	25	32
10	100	1024
20	400	1048576
30	900	1073741824

Fun, but counter-intuitive problem involving $f(x) = 2^x$. If 500 sheets of paper are 2 inches thick, if you folded a piece of paper 50 times, how thick would it be? (I know you can't really fold it 50 times, but if you could…)

Properties of exponentials

The first thing you should notice is that all functions of the form $f(x) = a^x$ pass through the point (0,1). This is because $a^0 = 1 \ \forall a$.

Here are some more properties:

$$a^{x+y} = a^x a^y \qquad a^{x-y} = \frac{a^x}{a^y}$$

$$(a^x)^y = a^{xy} \qquad (ab)^x = a^x b^x$$

$$\text{If } a > 1, \text{ then } \lim_{x\to\infty} a^x = \infty \text{ and } \lim_{x\to-\infty} a^x = 0$$

$$\text{If } 0 < a < 1, \text{ then } \lim_{x\to\infty} a^x = 0 \text{ and } \lim_{x\to-\infty} a^x = \infty$$

The Derivative of an exponential function

We will first attempt to find the derivative of the general exponential function $f(x)=a^x$. I'll warn you, we're going to hit a snag. Once we get "stuck" we will look at a particular exponential, namely $f(x)=e^x$. We will then get "unstuck" by looking at a special limit…and finally end up with the derivative of $f(x)=e^x$. Later on we'll go back to find the derivative of a general exponential $f(x)=a^x$.

If $f(x)=a^x$, then by the definition of derivative,

$$
\begin{aligned}
f'(x) &= \lim_{h\to o}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to o}\frac{a^{x+h}-a^x}{h} \\
&= \lim_{h\to o}\frac{a^x a^h-a^x}{h} \\
&= \lim_{h\to o}\frac{a^x(a^h-1)}{h} \\
&= a^x\lim_{h\to o}\frac{a^h-1}{h} \quad (\text{since } \lim_{h\to 0}a^x=a^x) \qquad \text{(I)}
\end{aligned}
$$

Now, note that if $f(x)=a^x$ and we wanted to find $f'(0)$, by definition of derivative,

$$
\begin{aligned}
f'(0) &= \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} \\
&= \lim_{h\to 0}\frac{a^{0+h}-a^0}{h} \\
&= \lim_{h\to 0}\frac{a^h-1}{h}
\end{aligned}
$$

Substituting this into (I) we obtain

$$f'(x)=a^x\, f'(0)$$

Now, this is both troublesome and interesting. Troublesome because it appears that in order to find the derivative of $f(x)=a^x$, we first need to find the value of its derivative at $x=0$! It's interesting because we have discovered an important fact…since $f'(0)$ is always a constant, <u>the derivative of an exponential function is the function itself times some constant</u>!

We've actually gone as far as we can now with the derivative of the general exponential function $f(x)=a^x$ and will now shift our attention to $f(x)=e^x$.

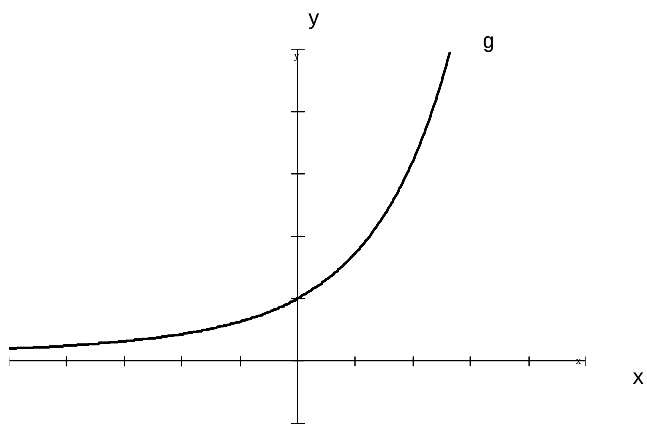
If we had started with trying to find the derivative of $f(x) = e^x$, we would have ended at the step where we had

$$f'(x) = a^x \lim_{h \to 0} \frac{a^h - 1}{h}$$

and for $f(x) = e^x$ this step would become

$$f'(x) = e^x \lim_{h \to 0} \frac{e^h - 1}{h} \qquad \text{(II)}$$

Let's take a look at the limit $\lim_{h \to 0} \frac{e^h - 1}{h}$. We know this is $f'(0)$ but we'd like to find out its value. Although we do not yet have the analytic techniques to evaluate this limit, we can explore it graphically. To graph, we will let $g(x) = \frac{e^x - 1}{x}$ and look to see what happens as $x \to 0$.



Even though g $\nexists$ at $x = 0$, it is clear that $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$. Substituting this limit value into (II) yields,

$$f'(x) = e^x$$

We know have the derivative of at least one exponential function.

If $f(x) = e^x$ then $f'(x) = e^x$.

And yes, it's everyone's favorite derivative!

Applying the Chain rule to this function gives us

If $g(x) = e^{f(x)}$ then $g'(x) = e^{f(x)} f'(x)$
or
$D_x[e^{f(x)}] = f'(x)\, e^{f(x)}$

Example 1

If $f(x) = e^{7x}$, find $f'(x)$.

$$f'(x) = 7e^{7x}$$

Example 2

If $f(x) = e^{\sin 8x}$, find $f'(x)$.

$$\begin{aligned} f'(x) &= e^{\sin 8x}(\cos 8x)8 \\ &= 8e^{\sin 8x}\cos 8x \end{aligned}$$

Example 3

If $f(x) = xe^{x}$, find $f'(x)$.

$$\begin{aligned} f'(x) &= (x)(e^{x}) + (e^{x})(1) \\ &= xe^{x} + e^{x} \\ &= e^{x}(x+1) \end{aligned}$$

Example 4

If $f(x) = xe^{-x^2}$, find $f'(x)$.

$$\begin{aligned} f'(x) &= (x)(-2xe^{-x^2}) + (e^{-x^2})(1) \\ &= -2x^2e^{-x^2} + e^{-x^2} \\ &= e^{-x^2}\left(1-2x^2\right) \end{aligned}$$

Example 5

Find an equation of a tangent to $2e^{xy} = x + y$ at $(0,2)$.

Point	Slope	Equation of tangent
Given (0,2)	$2e^{xy} = x + y$	$y - 2 = 3(x - 0)$
	$2e^{xy}\left(x\frac{dy}{dx} + y\right) = 1 + \frac{dy}{dx}$	
	$\frac{dy}{dx} = \frac{1-2ye^{xy}}{2xe^{xy}-1}$	
	$\left.\frac{dy}{dx}\right\vert_{(0,2)} = 3 \quad \therefore m_T = 3$	

Limits involving e^x as *x* ® ¥ and *x* ® - ¥

Many texts treat these limits differently, I suggest you use your common sense. Do things like eliminate negative exponents, get common denominators and then think of *x* as a "really big number."

Example 6

$$\lim_{x\to\infty}\frac{e^{3x}-e^{-3x}}{e^{3x}+e^{-3x}}$$

$$\lim_{x\to\infty}\frac{e^{3x}-e^{-3x}}{e^{3x}+e^{-3x}}=\lim_{x\to\infty}\frac{e^{3x}-\frac{1}{e^{3x}}}{e^{3x}+\frac{1}{e^{3x}}}$$

$$=\lim_{x\to\infty}\frac{e^{3x}-\frac{1}{e^{3x}}}{e^{3x}+\frac{1}{e^{3x}}}\cdot\frac{e^{3x}}{e^{3x}}$$

$$=\lim_{x\to\infty}\frac{e^{6x}-1}{e^{6x}+1}$$

Now, the -1 and $+1$ really do not matter, we have a "really big number" over the same "really big number" and we get

$$\lim_{x\to\infty}\frac{e^{3x}-e^{-3x}}{e^{3x}+e^{-3x}}=1$$

Example 7

$$\lim_{x\to1^+}e^{\frac{2}{x-1}}$$

Think of $e^{\frac{2}{x-1}}$ as $e^{2\frac{1}{x-1}}$. Now, as $x\to1^+$, $\frac{1}{x-1}\to\infty$ so we get $e^{2\infty}$

$$\therefore\lim_{x\to1^+}e^{\frac{2}{x-1}}=\infty\ \text{(does not exist)}$$

Logarithmic Functions and their Derivatives

Introduction

Logarithmic functions are functions of the form $f(x) = \log_a x$ or $y = \log_a x$. The logarithm of a number *x* is simply the exponent to which a base *a* must be raised to obtain *x*. For example, $\log_2 8 = 3$ because $2^3 = 8$. This gives us the definition:

$$\log_a x = y \text{ iff } a^y = x$$

Example 1

Find $\log_3 81$

$$\log_3 81 = x$$
$$3^x = 81$$
$$x = 4$$

Example 2

Solve $\log_{25} 5 = x$

$$\log_{25} 5 = x$$
$$25^x = 5$$
$$\left(5^2\right)^x = 5$$
$$5^{2x} = 5^1$$
$$2x = 1$$
$$x = \frac{1}{2}$$

Properties of logarithms

You are probably familiar with these basic properties of logarithms…

$$\log_a xy = \log_a x + \log_a y$$

$$\log_a x^y = y \log_a x$$

$$\log_a \frac{x}{y} = \log_a x - \log_a y$$

It was these properties of logarithms which made them so useful when they were invented by Napier in the first half of the 17th century. Logarithms basically allow us to change multiplication problems to addition, division problems to subtraction and exponentiation to multiplication. This was a boon to astronomers of the day who, prior to logarithms, spent the majority of their time in tedious calculations.

We now introduce two more properties of logarithms.

Consider $\log_a a^x$. This is basically asking "To what value must a be raised to obtain a^x?" Of course, the answer is "x", thus

$$\log_a a^x = x$$

Now consider $a^{\log a^x}$. We know that, based on the definition of logarithm, if $y = \log a^x$ then $a^y = x$ and and thus

$$a^{\log_a x} = x$$

Important note: The logarithmic function $f(x) = \log_a x$ has a domain of $(0,\infty)$, a range of $(-\infty,\infty)$, is always increasing and is continuous and differentiable everywhere in its domain.

As you have already guessed from the properties above, the logarithmic and exponential functions are inverses of each other.

The natural logarithmic function

Just as the most important exponential function for us is the natural exponential function $f(x) = e^x$, the most important logarithmic function, for us, is the natural logarithmic function. The natural exponential function is a base e logarithm, $f(x) = \log_e x$. This function is used so often and is of such import to mathematicians that it has earned it's own notation, $f(x) = \ln x$. The properties listed above in terms of $\ln x$ are:

$$\ln x = y \ \ iff \ \ e^y = x$$

$$\ln e^x = x$$

$$e^{\ln x} = x$$

When you think of the last two of these properties, it may be more helpful to think of them in these terms:

$$\ln e^{\clubsuit} = \clubsuit \quad \textit{and} \quad e^{\ln \clubsuit} = \clubsuit$$

Applying these properties we see that,

$$\ln e^{\sin x} = \sin x$$
$$\ln e^{2x-7} = 2x-7$$

$$e^{\ln \cos x} = \cos x$$
$$e^{\ln(8x+3)} = 8x+3$$

One final property which is very useful…

Any base logarithm can be expressed in terms of the natural logarithm. This is immediately useful since your calculator can only operate with base 10 (common logs) and base *e* (natural logs)!

$$\text{If } \; y = \log_a x$$

$$\text{then } \; a^y = x$$

Taking the natural logarithm of both sides

$$\ln a^y = \ln x$$
$$y \ln a = \ln x$$
$$y = \frac{\ln x}{\ln a}$$

But $y = \log_a x$ so we get

$$\log_a x = \frac{\ln x}{\ln a}$$

Derivative of logarithmic functions

We will first concern ourselves with the derivative of $f(x) = \ln x$. After we have that derivative, we will discuss the derivative of logarithmic functions with other bases.

We start with

$$y = \ln x$$

Which, by definition, means

$$e^y = x$$

Implicitly differentiating yields

$$e^y \frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \frac{1}{e^y}$$

But we already know $e^y = x$ so

$$\frac{dy}{dx} = \frac{1}{x}$$

We now have

If $f(x) = \ln x$ *then* $f'(x) = \frac{1}{x}$

Applying the chain rule,

$$D_x[\ln f(x)] = \frac{1}{f(x)} f'(x)$$

or

$$D_x[\ln f(x)] = \frac{f'(x)}{f(x)}$$

Example 3

If $f(x) = \ln(\sin x)$ find $f'(x)$.

$$f'(x) = \frac{1}{\sin x}\cos x$$
$$= \frac{\cos x}{\sin x}$$
$$= \cot x$$

Example 4

If $f(x) = \ln \dfrac{x+1}{\sqrt{x-2}}$ find $f'(x)$.

First applying the properties of logarithms (VERY IMPORTANT) we get

$$f(x) = \ln(x+1) - \ln\sqrt{x-2}$$
$$f'(x) = \frac{1}{x+1}(1) - \frac{1}{\sqrt{x-2}}\left(\frac{1}{2}\right)(x-2)^{-\frac{1}{2}}(1)$$
$$= \frac{1}{x+1} - \frac{1}{2(x-2)}$$

Example 5

If $f(x) = \ln(\ln x)$ find $f'(x)$.

$$f'(x) = \frac{1}{\ln x}\frac{1}{x}$$
$$= \frac{1}{x\ln x}$$

Derivatives of general logarithmic functions

We need to find the derivative of $f(x) = \log_a x$.

Since $\log_a x = \dfrac{\ln x}{\ln a}$ we can say

$$D_x[\log_a x] = D_x\left[\frac{\ln x}{\ln a}\right]$$

Since $\dfrac{1}{\ln a}$ is a constant,

$$D_x[\log_a x] = \frac{1}{\ln a} D_x[\ln x]$$
$$= \frac{1}{\ln a}\frac{1}{x}$$
$$= \frac{1}{x\ln a}$$

Applying the chain rule,

$$D_x[\log_a f(x)] = \frac{f'(x)}{f(x)\ln a}$$

The only "problem" with this form is that regardless of the base which is in the problem, the answer is given in terms of the natural logarithm. There are times when we want to give an answer that is in the same base as the original problem.

We know that $\log_a x = \frac{\ln x}{\ln a}$.

Letting $x = e$, we get

$$\log_a e = \frac{\ln e}{\ln a}$$
$$= \frac{1}{\ln a}$$

Since $\log_a e = \frac{1}{\ln a}$ we can eliminate the $\frac{1}{\ln a}$ from the derivative and obtain,

$$D_x[\log_a f(x)] = \frac{f'(x)}{f(x)\ln a} = \frac{f'(x)\log_a e}{f(x)}$$

Example 6

If $f(x) = \log_9(2+\sin x)$ find $f'(x)$.

$$f'(x) = \frac{\cos x}{(2+\sin x)\ln 9}$$

or

$$= \frac{(\cos x)\log_9 e}{2+\sin x}$$

Derivative of the general exponential function

Now that we have the derivative of logarithmic functions, we can use them to help us derive the derivative of the general exponential function. We already know the derivative of $f(x)=e^x$, and now its time to find a derivative for $f(x)=a^x$.

We want to find $D_x[a^x]$.

We know that $a^x = e^{\ln a^x} = e^{x\ln a}$ so

$$\begin{aligned} D_x[a^x] &= D_x[e^{x\ln a}] \\ &= e^{x\ln a} D_x[x\ln a] \end{aligned}$$

Now, $\ln a$ is a constant, so we bring it outside of the derivative,

$$D_x[a^x] = e^{x\ln a}(\ln a)1$$

We also know that $e^{x\ln a} = e^{\ln a^x} = a^x$. Substituting yields,

$$D_x[a^x] = a^x \ln a$$

Applying the chain rule finally gives us what we wanted,

$$D_x[a^{f(x)}] = a^{f(x)}(\ln a)(f'(x))$$

or

$$D_x[a^{f(x)}] = a^{f(x)} f'(x)\ln a$$

Example 7

If $f(x)=10^{x^2}$ find $f'(x)$.

$$f'(x) = \left(10^{x^2}\right)(2x)(\ln 10)$$

Logarithmic differentiation

We can now find the derivative of a function raised to a number (the power rule) and the derivative of a number raised to a function (general exponentials). How would we address a problem in which a function is raised to a function? We use a technique called "logarithmic differentiation". It is not a technique we use often. It is used to simplify differentiation of complex functions and functions where we have $h(x) = f(x)^{g(x)}$.

Consider $f(x) = x^{x^2}$. Let's use logarithmic differentiation to find $f'(x)$.

First, let $y = f(x)$

$$y = x^{x^2}$$

Now take the natural log of both sides

$$\ln y = \ln x^{x^2}$$

Using the properties of logarithms we get

$$\ln y = x^2 \ln x$$

Now, implicitly differentiating yields

$$\frac{1}{y}\frac{dy}{dx} = (x^2)\frac{1}{x} + (\ln x)(2x)$$

Solving for $\frac{dy}{dx}$

$$\frac{dy}{dx} = y\left(x + 2x\ln x\right)$$

But we know that $y = x^{x^2}$ so,

$$\frac{dy}{dx} = x^{x^2}\left(x + 2x\ln x\right)$$

Example 8

If $f(x) = x^{\sqrt{x}}$, find $f'(x)$.

Let $y = f(x)$

$$y = x^{\sqrt{x}}$$
$$\ln y = \ln x^{\sqrt{x}}$$
$$\ln y = \sqrt{x}\ln x$$
$$\frac{1}{y}\frac{dy}{dx} = \sqrt{x}\frac{1}{x} + \ln x\frac{1}{2\sqrt{x}}$$
$$\frac{dy}{dx} = x^{\sqrt{x}}\left(\frac{\sqrt{x}}{x} + \frac{\ln x}{2\sqrt{x}}\right)$$

Exponential Growth

Introduction

Exponential growth is based on direct variation. Direct variation is a relationship in which the dependent quantity varies as a factor of the independent variable. This is expressed as $y = kx$. In exponential growth, the growth rate of a quantity is directly proportional to the amount present at some initial time. This relationship can be expressed:

$$\frac{dy}{dt} = ky$$

where y is the amount present, $\frac{dy}{dt}$ is the growth rateof y over time and k is the growth constant.

Now, we want a function which will tell us how much of a quantity is present after some period of time. What we have now is the derivative $\frac{dy}{dt}$. The equation $\frac{dy}{dt} = ky$ is what we call a differential equation. It's an equation with a derivative in it. To solve differential equations, we need to "antidifferentiate". Since we do not yet know how to antidifferentiate, we will have to find another way.

Starting with $\frac{dy}{dt} = ky$, divide both sides by y to get

$$\frac{1}{y}\frac{dy}{dt} = k \quad \text{(remember that } k \text{ is a constant)}$$

Now, multiply both sides by *dt*

$$\frac{1}{y}dy = k\,dt$$

This equation is what we would get had we started with the equation $\ln y = kt + C$ and differentiated implicitly with respect to *t*! Because this is true, we can take the next step and say

$$\frac{1}{y}dy = k\,dt$$

is equivalent to

$$\ln y = kt + C$$

The "C" is there because we could differentiate all of the following equations implicitly with respect to t and still get $\frac{1}{y}dy = k\,dt$.

$$\ln y = kt + 7$$
$$\ln y = kt - 9$$
$$\ln y = kt + 23$$

During the differentiation, the constant on the end goes to zero. When we work backward, we have to include the "C" to make sure our "antiderivative" is all-inclusive. We talk more about this later…right now just know that it has to be there!

So we have

$$\ln y = kt + C$$

which is equivalent to

$$y = e^{kt+C}$$

Using properties of exponents,

$$y = e^{kt}e^{C}$$

Now, e^C is just another constant so we can rewrite our equation as

$$y = Ae^{kt}$$

One more step to complete our model for exponential growth. When $t = 0$ $y = y_o$ where y_o represents the initial amount present. Substituting these values yeilds

$$y_o = Ae^0$$

which means that the "*A*" is the initial amount of material present.

This all results in our model for exponential growth:

$$\boxed{y = y_o e^{kt}}$$

In this model,

- y is the end amount of material present after time t
- y_o is the initial amount of material present
- k is the growth rate
- t is time

When this model is applied to problems involving money it is usually expressed:

$$A = Pe^{rt}$$

In this model

A is the amount of money after time t
P is the principle, the initial investment
r is the interest rate (the growth rate)
t is time

Types of problems

We normally face two types of problems involving exponential growth.

In the easier of the two types, we are given a data point—usually a time and an amount present—and the growth rate. These problems usually done by simply substituting into our model to determine some amount after a period of time.

In the second type of problem we are given two data points. In this case, we use the data points to find the growth constant, k, and an initial amount. This specifies the model to our problem. We then use our model to answer questions.

Example 1

In a certain culture, the rate of growth of bacteria is proportional to the amount present. If 1000 bacteria are present initially and the amount doubles in 12 minutes, how long will it be before there will be one million bacteria?

We have been given two data points. When $t = 0,\ y = 1000$ and when $t = 12,\ y = 2000$. Using this information we can say that $y_o = 1000$ and find ***k***.

$$2000 = 1000e^{12k}$$

or

$$2 = e^{12k}$$

Solving for t by taking the natural log of both sides, we get

$$\ln 2 = \ln e^{12k}$$
$$\ln 2 = 12k$$
$$\frac{\ln 2}{12} = k$$

Our model now specified to our problem is

$$y = 1000e^{\frac{\ln 2}{12}t}$$

To determine when there will be one million bacteria, let *y* = 1,000,000 and solve for *t*.

$$1,000,000 = 1000e^{\frac{\ln 2}{12}t}$$
$$1,000,000 = 1000e^{\frac{\ln 2}{12}t}$$
$$\ln 1000 = \ln e^{\frac{\ln 2}{12}t}$$
$$\ln 1000 = \frac{\ln 2}{12}t$$
$$t = \frac{\ln 1000}{\frac{\ln 2}{12}}$$
$$t \approx 119.589$$

$\therefore$ there will be one million bacteria present after 119.589 minutes.

Note: When you get an expression for k, do not use it's three decimal approximation. Instead, store the exact value, like $\frac{\ln 2}{12}$, in some variable. Then use this variable when making a final calculation.

Example 2

The rate of increase in the population of a certain city is proportional to the population. If the population in 1950 was 50,000 and in 1980 it was 75,000, what will the population be in 2010?

We have two data points. When $t = 0$, $y = 50,000$ and when $t = 30$, $y = 75,000$. We use $y_o = 50,000$ and write

$$75,000 = 50,000e^{30k}$$

Remember, our first objective is to find *k*, the growth rate.

Solving this for k (using natural logs) we obtain

$$k = \frac{1}{30}\ln\frac{3}{2}$$

Our model now becomes

$$y = 50,000e^{\left(\frac{1}{30}\ln\frac{3}{2}\right)t}$$

We need the population in 2010 which means t = 60. (We could also use t = 30 if we use y_o = 75000).

$$y = 50,000e^{\left(\frac{1}{30}\ln\frac{3}{2}\right)60}$$
$$y = 112,500$$

Thus the population in 2010 will be 112,500.

Example 3 Decay

(Decay problems are handled in the same manner—we will simply have a negative growth rate.)

The rate of decay of radium is proportional to the amount present at any time. If 60 mg of radium are present now and its half-life is 1690 years, how much radium will be present 100 years from now?

$$\text{When } t = 0,\ y = 60 \text{ so } y_o = 60$$
$$\text{When } t = 1690,\ y = 30$$

$$\text{So, } 30 = 60e^{1690k}$$
$$1 = 2e^{1690k}$$
$$\frac{1}{2} = e^{1690k}$$
$$\ln\frac{1}{2} = \ln e^{1690k}$$
$$\ln 1 - \ln 2 = 1690k$$
$$k = -\frac{\ln 2}{1690}$$

$$\therefore\ y = 60e^{-\left(\frac{\ln 2}{1690}\right)t}$$

To determine how much is left after 100 years, let $t = 100$

$$y = 60e^{-\left(\frac{\ln 2}{1690}\right)100}$$

$$y \approx 57.589$$

$\therefore$ There will be approximately 57.589 mg of radium left after 100 years.

Half-life and doubling time

We can easily derive a simple formula to determine the half-life or doubling time of a given quantity undergoing exponential growth.

To determine half-life let $y = \frac{1}{2}$ and $y_o = 1$

$$y = y_o e^{kt}$$

$$\frac{1}{2} = 1e^{kt}$$
$$\ln\frac{1}{2} = \ln e^{kt}$$
$$\ln 1 - \ln 2 = kt$$

∴ Half-life is

$$t = -\frac{\ln 2}{k}$$

To determine the doubling time (mostly used with money problems when you are asked to determine how long it takes a given amount of money invested at a given interest rate to double):

Let $y = 2$ *and* $y_o = 1$

$$2 = e^{kt}$$
$$\ln 2 = kt$$

∴ Doubling time is

$$t = \frac{\ln 2}{k}$$

Note that both expressions are the same except for the negative. This comes about because when we speak of half-life, the rate of growth is negative.

The Inverse Trigonometric Functions

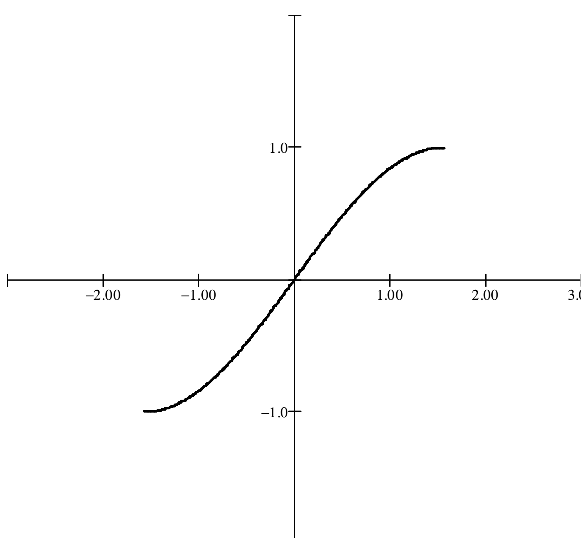
Introduction

We would be remiss if we had an entire discussion of inverse functions and left out the six trigonometric functions. The problem is, none of them are monotonic (one-to-one) and thus cannot have inverses. Well, we can "fix" the trigonometric functions by restricting their domains—making them one-to-one. It's one of those "things" mathematicians do when we want something that appears to contradict what we already have…we change the problem.

By the way, you will see two different notations for the inverse trigonometric functions. Inverse sine, for example, can be written as $\sin^{-1} x$ or as arcsin *x*. The arc notation actually gets more to the notion of how all trigonometric functions are defined. For instance, if we wanted to find an angle whose sine is $\frac{1}{2}$ we could ask for $\sin^{-1}\frac{1}{2}$ or we could ask for $\arcsin\frac{1}{2}$. The former notation (with the -1) certainly fits our inverse notation better than "arcsin"…after all, we call f^{-1} the inverse of *f*. The arcsin notation however, asks us to <u>find the length of the arc</u> whose sine is $\frac{1}{2}$. Since we measure angles in radians and a radian is a measure of a length of arc subtended by an angle, the arcsin notation is more descriptive of what we are actually doing.

Inverse sine ($\arcsin x$ or $\sin^{-1} x$)

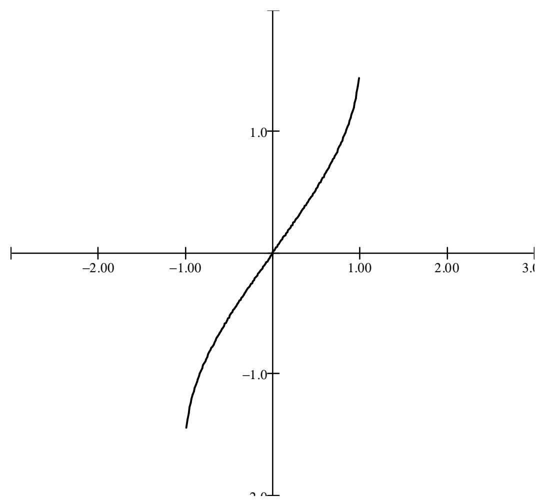
To get the inverse for the sine function, we restrict the domain. Our new sine function will be restricted to the domain $\left[-\frac{p}{2},\frac{p}{2}\right]$. The range of our new sine function is still $[-1,1]$. Our new sine function is graphed below.



This means that

- the domain of $\sin^{-1} x$ is [-1,1]
- the range of $\sin^{-1} x$ is $\left[-\frac{p}{2}, \frac{p}{2}\right]$

The graph of $y = \sin^{-1} x$ is



Definition: $y = \sin^{-1} x$ iff $x = \sin y$ where $y \in \left[-\frac{p}{2}, \frac{p}{2}\right]$

This means that an answer to a question that asks $\sin^{-1}(\textit{number})$ must be in $\left[-\frac{p}{2}, \frac{p}{2}\right]$.
Since sine and arcsine are inverses,

$$\sin(\sin^{-1} x) = x \quad \text{for} \quad x \in [-1,1]$$
and
$$\sin^{-1}(\sin x) = x \quad \textit{for} \quad x \in \left[-\frac{p}{2}, \frac{p}{2}\right]$$

Example 1

Find $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

Draw the appropriate reference angles. There are two of them, one in the first and one in the second quadrant. Since the answer must lie in $\left[-\frac{p}{2},\frac{p}{2}\right]$, the reference angle in the first quadrant is the one we want. Thus $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)=\frac{p}{3}$.

Example 2

Find $\sin^{-1}\left(\sin\frac{p}{12}\right)$

Since $\frac{p}{12}\in\left[-\frac{p}{2},\frac{p}{2}\right]$, $\sin^{-1}\left(\sin\frac{p}{12}\right)=\frac{p}{12}$

Example 3

Find $\sin\left(\sin^{-1}\frac{1}{\sqrt{2}}\right)$

Since $\frac{1}{\sqrt{2}}\in[-1,1]$, $\sin\left(\sin^{-1}\frac{1}{\sqrt{2}}\right)=\frac{1}{\sqrt{2}}$

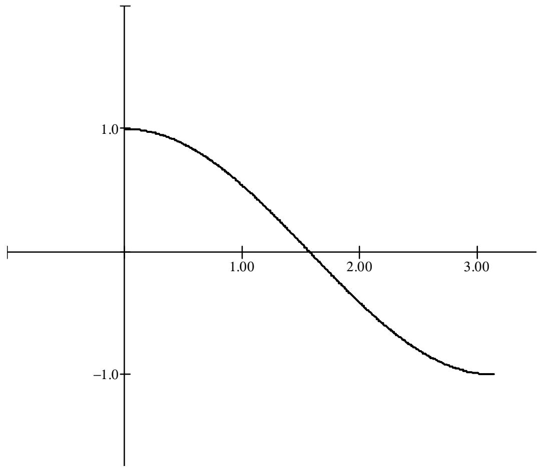
Example 4

Find $\sin^{-1}\left(\sin\frac{7p}{6}\right)$

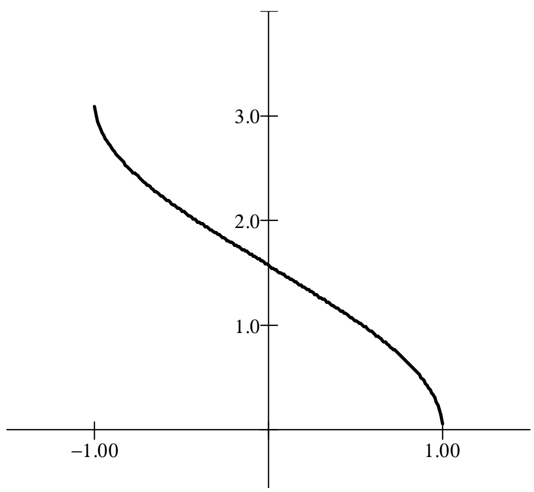
Since $\frac{7p}{6}$ is not in $\left[-\frac{p}{2},\frac{p}{2}\right]$, we have to draw the reference angle and find $\sin\frac{7p}{6}$ first. We then have $\sin^{-1}\left(-\frac{1}{2}\right)$. This requires two reference angles, one in Quadrant III and one in Quadrant IV. The one we need is in Quadrant IV since our answer must be in $\left[-\frac{p}{2},\frac{p}{2}\right]$. Therefore, $\sin^{-1}\left(\sin\frac{7p}{6}\right)=-\frac{p}{6}$.

Inverse cosine (arccos x or $\cos^{-1} x$)

In order to make cosine monotonic, we restrict its domain to [0,π]. The new range of cosine is still [-1,1].



The domain of $\cos^{-1} x$ is then [-1,1] and its range is [0,π].



Definition: $y = \cos^{-1} x$ iff $x = \cos y$ where $y \in \left[0, p\right]$.

This means that an answer to a question that asks $\cos^{-1}(\textit{number})$ must be in $[0, p]$.
Since cosine and arccosine are inverses,

$$\cos(\cos^{-1} x) = x \text{ for } x \in [-1,1]$$

and

$$\cos^{-1}(\cos x) = x \quad \textit{for} \quad x \in [0, p]$$

Example 5

Find $\cos^{-1} 0$

We are simply being asked to find what angle has a cosine of 0. Remember, it must be in $[0,\pi]$. Using the unit circle we determine that $\cos^{-1} 0 = \frac{\pi}{2}$.

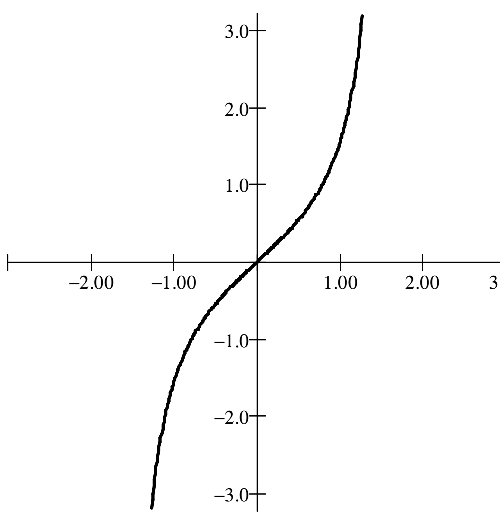
Example 6

Find $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

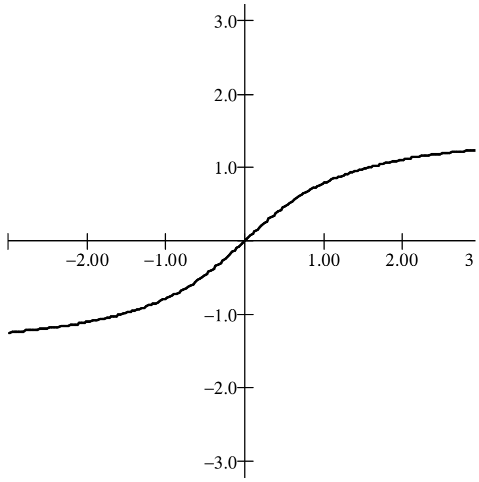
Draw the two reference angles. They will be in Quadrant II and Quadrant III. The one we need must be in Quadrant II, thus $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$.

Inverse tangent ($\tan^{-1} x$ *or* $\arctan x$)

To get at $\tan^{-1} x$ we restrict the domain of $\tan x$ to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. The range of our new tangent function is $(-\infty, \infty)$.



The domain of $\tan^{-1} x$ is then $(-\infty, \infty)$ and its range is $\left(-\frac{p}{2}, \frac{p}{2}\right)$. Below is the graph of $\tan^{-1} x$.



Definition: $y = \tan^{-1} x$ *iff* $x = \tan y$ *where* $y \in \left(-\frac{p}{2}, \frac{p}{2}\right)$

Example 7

Find $\tan^{-1} \sqrt{3}$

Since the tangent of an angle is the opposite over the adjacent, we draw the reference angles, one with an opposite side of $\sqrt{3}$ and an adjacent side of 1 and the other with an opposite side of $-\sqrt{3}$ and an adjacent side of -1. The first angle lies in Quadrant I and the second lies in Quadrant III. Since our answer must be in $\left(-\frac{p}{2}, \frac{p}{2}\right)$, we choose the angle in Quadrant I. Thus $\tan^{-1} \sqrt{3} = \frac{p}{3}$.

Example 8

Find $\tan^{-1} 0$

We are being asked to find an angle whose tangent is zero. Using the unit circle and the fact that the tangent is sine over cosine, $\tan^{-1} 0 = 0$.

Inverse cosecant, secant and cotangent

To be honest, we rarely use these inverses at this level of mathematics. We will learn their derivatives and you will have to know them, but we rarely face a problem like finding $\cot^{-1}\frac{1}{2}$. It's not that it is too difficult or complicated, but we just do not encounter them very often. In fact, some Calculus texts (like ours) skip them entirely and others define them differently! So, without loss of detail, we will define the last three inverse trig functions. Cotangent, cosecant and secant have strangely restricted domains to make them one-to-one.

> Definition: $y=\csc^{-1}x$ *iff* $x=\csc y$ *where* $y\in\left(-\pi,-\frac{\pi}{2}\right]\cup\left[0,\frac{\pi}{2}\right)$

The domain of $\csc^{-1}x$ is $|x|\geq 1$.

The range of $\csc^{-1}x$ is $\left(-\pi,-\frac{\pi}{2}\right]\cup\left[0,\frac{\pi}{2}\right)$.

> Definition: $y=\sec^{-1}x$ *iff* $x=\sec y$ *where* $y\in\left(0,\frac{\pi}{2}\right]\cup\left(\pi,\frac{3\pi}{2}\right]$

The domain of $\sec^{-1}x$ is $|x|\geq 1$.

The range of $\sec^{-1}x$ is $\left(0,\frac{\pi}{2}\right]\cup\left(\pi,\frac{3\pi}{2}\right]$.

> Definition: $y=\cot^{-1}x$ iff $x=\cot y$ where $y\in(0,\pi)$

The domain of $\cot^{-1}x$ is $(-\infty, \infty)$.

The range of $\cot^{-1}x$ is $(0, \pi)$.

The Derivatives of the Inverse Trigonometric Functions

Inverse sine

We begin with the two statements which define inverse sine,

$$y = \sin^{-1} x$$

equivalently,

$$x = \sin y$$

Differentiating implicitly yields,

$$1 = \cos y \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\cos y} \qquad (1)$$

This is the derivative, but it is in terms of *y*, not *x*. We want the derivative in terms of *x*.
Now, we have a trigonometric identity which can be written $\sin^2 y + \cos^2 y = 1$. We also know that $x = \sin y$ so $x^2 = \sin^2 y$. Substituting this last statement into the identity yields,

$$x^2 + \cos^2 y = 1$$

Solving for cos *y*,

$$\cos y = \sqrt{1 - x^2}$$

Now, substituting this into (1) we get

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

Applying the chain rule to this we have,

$$D_x[\sin^{-1} f(x)] = \frac{f'(x)}{\sqrt{1 - [f(x)]^2}}$$

Inverse cosine

Again, we begin with the two statements which define inverse cosine.

$$y = \cos^{-1} x$$
$$x = \cos y$$

Differentiating yields,

$$1 = -\sin y \frac{dy}{dx}$$

or

$$\frac{dy}{dx} = -\frac{1}{\sin y} \qquad (2)$$

We will get this derivative in terms of x using the same technique we used for inverse sine. Similar to the previous derivation, we know $x = \cos y \rightarrow x^2 = \cos^2 y$ and $\sin^2 y + \cos^2 y = 1$. Substituting yields

$$\sin^2 y + x^2 = 1$$

Solving for $\sin y$ gives us,

$$\sin y = \sqrt{1 - x^2}$$

We can now replace $\sin y$ in equation (2), which results in

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1 - x^2}}$$

Applying the chain rule we can say,

$$D_x[\cos^{-1} f(x)] = -\frac{f'(x)}{\sqrt{1 - [f(x)]^2}}$$

Notice that the only difference in the derivatives of inverse sine and inverse cosine is the negative!

Inverse tangent

The same process is used to derive the derivative of inverse tangent as was used for inverse sine and inverse cosine.

$$y = \tan^{-1} x$$

$$\text{so, } x = \tan y$$

Differentiating implicitly,

$$1 = \sec^2 y \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} \qquad (3)$$

Again, we have the derivative in terms of y. Using the identity $\sec^2 y = 1 + \tan^2 y$ and the fact that $x = \tan y \rightarrow x^2 = \tan^2 y$ we can say,

$$\sec^2 y = 1 + x^2$$

Substituting into (3) yields,

$$\frac{dy}{dx} = \frac{1}{1 + x^2}$$

Applying the chain rule,

$$D_x[\tan^{-1} f(x)] = \frac{f'(x)}{1 + [f(x)]^2}$$

Inverse secant

$$y = \sec^{-1} x$$
$$\text{so}, x = \sec y$$

Differentiating implicitly,

$$1 = \sec y \tan y \, \frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{1}{\sec y \tan y} \qquad (4)$$

We need to make two substitutions now to get this derivative in terms of *x*.

For the sec y, we use the previous statement that $x = \sec y$. Substituting into (4) gives us

$$\frac{dy}{dx} = \frac{1}{x \tan y}$$

Now, for tan y, we use the identity, $\tan^2 y = \sec^2 y - 1$ and the statement that $x = \sec y \to x^2 = \sec^2 y$. We can change the identity to $\tan^2 y = x^2 - 1$ which leads us to $\tan y = \sqrt{x^2 - 1}$. Substituting this last statement into (4) gives us

$$\frac{dy}{dx} = \frac{1}{x\sqrt{x^2 - 1}}$$

Applying the chain rule yields,

$$D_x[\sec^{-1} f(x)] = \frac{f'(x)}{f(x)\sqrt{[f(x)]^2 - 1}}$$

Inverse cosecant and inverse cotangent

These derivatives are obtained in a similar fashion to the four already derived.

$$D_x[\csc^{-1} f(x)] = -\frac{f'(x)}{f(x)\sqrt{[f(x)]^2 - 1}}$$

$$D_x[\cot^{-1} f(x)] = -\frac{f'(x)}{1 + [f(x)]^2}$$

Note: The derivative of inverse cosine is the negative of the derivative of inverse sine. The derivative of inverse cotangent is the negative of the derivative of inverse tangent. The derivative of inverse cosecant is the negative of the derivative of inverse secant.