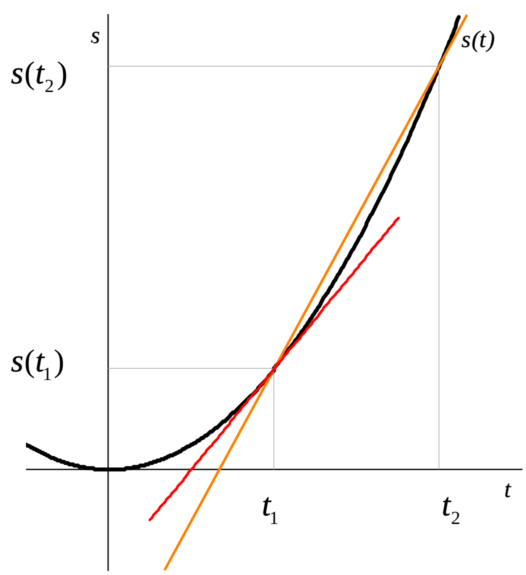
Rectilinear Motion

Introduction

One of the major reasons calculus was invented was the study of motion. Scientists and engineers were interested in being able to calculate not only the average velocity of an object, but also its instantaneous velocity. In fact, Newton's calculus was very motion-oriented and the analysis of motion was one of his prime interests. This emphasis on motion can even be seen in his choice of terms. Newton termed changing quantities *fluents* and their rates of change *fluxions*. A *fluxion* is very close to what we now call a derivative. The study of rectilinear motion is, then, a classic application of calculus.

Average velocity and instantaneous velocity

When we introduced the derivative, we spent some time on graphs of position functions. We learned that the slope of a tangent to the graph of a position function could be interpreted as instantaneous velocity. Consider the graph of a position function $s(t)$ below. The orange line is the secant line through the points $(t_1, s(t_1))$ and $(t_2, s(t_2))$. The red segment represents the tangent at t_1.



The slope of the secant line, given by $\frac{s(t_2)-s(t_1)}{t_2-t_1}$, represents the change in position divided by the change in time on the interval (t_1, t_2). This is the average velocity on (t_1, t_2). The slope of the tangent (the derivative) at t_1 represents the instantaneous velocity at t_1. It is $\lim_{t_2 \to t_1} \frac{s(t_2)-s(t_1)}{t_2-t_1}$. We can find the instantaneous velocity of an object by using the derivative.

$$s'(t) = v(t)$$
The derivative of position is velocity.

From now on, when we speak of "velocity" we are usually referring to instantaneous velocity, not average velocity.

In the course of some problems, our calculations may result in a negative velocity. The negative is an indication of direction. By convention, movement to the left or down is considered negative. Thus, if we are working on a problem about an object being thrown into the air, a positive velocity indicates the object is moving upward…a negative velocity indicates the object is moving downward. *Speed* is the absolute value of velocity.

Acceleration

We have determined that a change in position divided by a change in time is velocity. What happens when we divide a change in velocity by a change in time? If we are in a vehicle and our velocity is changing, we are accelerating. Thus, the average acceleration of an object can be given by:

$$\text{average acceleration} = \frac{v(t_2) - v(t_1)}{t_2 - t_1}.$$

If we take the limit of this different quotient as $t_2 \to t_1$, we would have the instantaneous acceleration. Again, when we use the term acceleration in this course, we are normally referring to instantaneous acceleration.

If $s(t)$ is a function that describes the position of an object at any time t,

$$s'(t) = v(t)$$

$$s''(t) = v'(t) = a(t)$$

In order to understand the relationship between velocity, speed and acceleration it may be helpful if we thing of acceleration as a force. If for instance, an object has a positive velocity (moving to the right) and a negative acceleration (pushed to the left), its speed will be decreasing. The table below illustrates the relationships.

If $v > 0$ and $a > 0$ the object is moving to the right and being pushed to the right. Its velocity is increasing and its speed is increasing.

If $v > 0$ and $a < 0$ the object is moving to the right and being pushed to the left. Its velocity is decreasing and its speed is decreasing.

If $v < 0$ and $a > 0$ the object is moving to the left and being pushed to the right. Its velocity is increasing (becoming less negative!) and its speed is decreasing.

If $v < 0$ and $a < 0$ the object is moving to the left and being pushed to the left. Its velocity is decreasing and its speed is increasing.

Note: We will see position functions written several ways. Many times, if an object is moving horizontally along the number line, the position function will be written as "$x(t) =$". Position functions can also be written as "$s(t) =$", "$s =$", "$y =$", or "$y(t) =$".

Example 1

A particle is moving so that its position is given by $s = 2t^3 - 4t^2 + 2t - 1$. Determine the intervals when the particle is moving to the right and the intervals when the particle is moving to the left. Determine when the particle is at rest and when it is changing direction.

All of these questions can be answered using a table. We begin by finding the derivative.

$$\frac{ds}{dt} = 6t^2 - 8t + 2$$

We need to determine when this derivative is positive, negative and zero. We use our normal sign chart method—first determining when the derivative is zero and considering its existence.

$$\frac{ds}{dt} = 0 \text{ when } 6t^2 - 8t + 2 = 0$$

$$3t^2 - 4t + 1 = 0$$

$$(3t-1)(t-1) = 0$$

$$t = \tfrac{1}{3} \text{ or } t = 1$$

$$\frac{ds}{dt} \text{ exists for all } t$$

	$\frac{ds}{dt}$
$(-\infty, 1/3)$	+
$t = 1/3$	0
$(1/3, 1)$	−
$t = 1$	0
$(1, \infty)$	+

From the table above we can conclude:

The particle is moving to the right on $(-\infty, 1/3) \cup (1, \infty)$.

The particle is moving to the left on $(1/3, 1)$.

The particle is at rest at $t = 1/3$ and $t = 1$.

The particle changes direction where the velocity changes sign…at $t = 1/3$ and $t = 1$.

Example 2

A ball is thrown upward from the ground with an initial velocity of 64 feet per second. If positive is up, the ball's position is given by $s(t) = -16t^2 + 64t$ where t is measured in seconds and position in feet.
(a) Is the ball rising or falling at $t = 1$?
(b) Is the ball rising or falling at $t = 3$?
(c) With what velocity does the ball strike the ground?
(d) How high does the ball go?

We begin all by finding $v(t)$.

$$v(t) = -32t + 64$$

(a) $v(1) = 32 \therefore$ the velocity at $t = 1$ is 32 feet per second. Because the velocity is positive, the ball is rising.

(b) $v(3) = -32 \therefore$ the velocity at $t = 3$ is -32 feet per second. Because the velocity is negative, the ball is falling.

(c) We must use the velocity function to determine the velocity at impact. Before we can determine the velocity when the ball hits the ground, we need to know when it hits the ground. We determine this by setting the position function equal to zero.

$$s(t) = -16t^2 + 64t$$

$$s(t) = 0 \text{ when } -16t^2 + 64t = 0$$

$$t = 0 \text{ or } t = 4$$

We now know that the ball hits the ground after 4 seconds.
Since $v(4) = -64$, the ball strikes the ground with a velocity of -64 feet per second…or 64 feet per second downward.

(d) When an object is thrown or launched into the air, its velocity at its maximum height is zero. We will use this fact to determine the maximum height. We will determine at what value of t the velocity is zero, then find the value of the position function at that t.

$$v(t) = -32t + 64$$

$$v(t) = 0 \text{ when } t = 2$$

Since $s(2) = 64$, the maximum height the ball reaches is 64 feet.

Implicit Differentiation

Introduction

Functions can be divided into two large categories—explicitly defined and implicitly defined. All the functions we have seen thus far in the course have been explicitly defined. We were always explicitly told what the dependent variable was in terms of the independent variable. Here are a few examples of explicitly defined functions:

$$f(x) = x^4 - 8x^2$$
$$q(z) = \sin z$$
$$y = 8x^2 + 9x$$
$$a = d^7 - 4d^2$$

In an implicitly defined function, we imply that one variable is a function of another but never state it outright. We can even have a situation where the variables in an equation are all functions of another variable not stated. Here are a few examples of implicitly defined functions (notice that we are not restricted to two variables):

$$x^2 + y^2 = z^2$$
$$xy - x^2y^3 = 9$$
$$a - 2ab + b^2 = a^3 + b^3$$

Consider the first example, $x^2 + y^2 = z^2$. In this equation the *x*, *y* and *z* could all vary over time and therefore be functions of *t*. It could be that $x = t^2$, $y = \sin t$ and $z = 2t$. If this were so, we could rewrite $x^2 + y^2 = z^2$ as $t^4 + \sin^2 t = 4t^2$. With implicit functions though, we are never told how any of the variables are defined. So how do we know what are the independent and dependent variables? The relationships are implied in the problem.

Consider the equation $x^3y^2 + 3y = 5$. If we were asked to differentiate this with respect to *x*, we know that *x* is the dependent variable and *y* is actually a function in *x*. If we were asked to differentiate with respect to *p*, we know that both the *x* and *y* are functions in *p* (maybe $x = 2p$ and $y = p^3$ …we don't know!)

To differentiate implicitly defined functions we use implicit differentiation. Implicit differentiation is an application of the chain rule. Let's examine several examples to see how it works.

Example 1

Determine $D_x[y^7]$.

We are being asked to differentiate y^7 with respect to *x*. What is being implied is that *y* is a function of *x*—but we're not being told the exact relationship. If we knew that $y = 8x + 1$, then y^7 would be $(8x+1)^7$. Differentiating would yield $7(8x+1)^6(8)$. <u>But we don't know what *y* is</u>!

Because y is a function of x, we could write our problem $D_x\left[[f(x)]^7\right]$. Applying the chain rule would result in $7[f(x)]^6 f'(x)$. Rewriting this in terms of y yields $7y^6 \frac{dy}{dx}$. Therefore $D_x[y^7] = 7y^6 \frac{dy}{dx}$.

Example 2

Determine $D_m[y^7]$.

The implication here is that y is a function of m therefore,

$$D_m[y^7] = 7y^6 \frac{dy}{dm}$$

(The $\frac{dy}{dm}$ is the derivative of y with respect to m.)

Example 3

If $x^2 + y^5 = 9$, find $\frac{dy}{dx}$.

Here the implication is that y is a function of x.
The derivative of x^2 with respect to x is $2x$.
The derivative of y^5 with respect to x is $5y^4 \frac{dy}{dx}$. You can think of y^5 as $[f(x)]^5$ and the derivative of $[f(x)]^5$ is $5[f(x)]^4 f'(x)$ so the derivative of y^5 is $5y^4 \frac{dy}{dx}$.
The derivative of 9 is zero.
Putting it all together yields

$$x^2 + y^5 = 9$$

$$2x + 5y^4 \frac{dy}{dx} = 0$$

$$5y^4 \frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -\frac{2x}{5y^4}$$

Example 4

Given $y^5 + 3x^2y^2 + 5x^4 = 12$, find $\frac{dy}{dx}$.

$$y^5 + 3x^2y^2 + 5x^4 = 12$$

$$5y^4 \frac{dy}{dx} + \left[(3x^2)\left(2y\frac{dy}{dx}\right) + (y^2)(6x)\right] + 20x^3 = 0$$

$$5y^4 \frac{dy}{dx} + 6x^2y\frac{dy}{dx} + 6xy^2 + 20x^3 = 0$$

$$\frac{dy}{dx}\left[5y^4 + 6x^2y\right] = -20x^3 - 6xy^2$$

$$\frac{dy}{dx} = -\frac{20x^3}{5y^4 + 6x^2y}$$

Note that in the first step, when we encountered the $3x^2y^2$ we used the product rule.

Example 5

Given $x^4 + y^3 = 8xy$, find $\dfrac{dy}{dx}$.

$$x^4 + y^3 = 8xy$$
$$4x^3 + 3y^2 \tfrac{dy}{dx} = (8x)\left(\tfrac{dy}{dx}\right) + (y)(8)$$
$$3y^2 \tfrac{dy}{dx} - 8x \tfrac{dy}{dx} = 8y - 4x^3$$
$$\tfrac{dy}{dx}\left[3y^2 - 8x\right] = 8y - 4x^3$$
$$\frac{dy}{dx} = \frac{8y - 4x^3}{3y^2 - 8x}$$

Example 6

Given $x = \sin y$, find $\dfrac{dy}{dx}$.

$$x = \sin y$$
$$1 = (\cos y)\tfrac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{1}{\cos y}$$
$$\frac{dy}{dx} = \sec y$$

Now that we have implicit differentiation, we never have to solve an equation like $x^2 + y = xy$ for y to be able to find $\frac{dy}{dx}$. We just implicitly differentiate. As a matter of fact, you should not attempt to solve for a variable before differentiation—it usually leads to trouble.

Example 7

Differentiate $x^3 + m^6 = 5w$ with respect to *t*.

$$x^3 + m^6 = 5w$$
$$3x^2 \tfrac{dx}{dt} + 6m^5 \tfrac{dm}{dt} = 5\tfrac{dw}{dt}$$

There is nothing left to do here…we have differentiated with respect to *t*.

Example 8

Given $x^4 + [f(x)]^5 = 9$, find $f'(x)$.

$$x^4 + [f(x)]^5 = 9$$
$$4x^3 + 5[f(x)]^4 f'(x) = 0$$
$$f'(x) = -\frac{4x^3}{5[f(x)]^4}$$

Example 9

Given $x^4 + y^5 = 9$, find $\frac{dy}{dx}$.

$$x^4 + y^5 = 9$$
$$4x^3 + 5y^4 \tfrac{dy}{dx} = 0$$
$$\frac{dy}{dx} = -\frac{4x^3}{5y^4}$$

Note that Examples 8 and 9 are the same problem written with different notation.

Related Rates

Introduction

Related rate problems are applications of implicit differentiation. They are one of the core of classic problems presented in a first year calculus course. No one can take calculus without encountering related rate problems. "Word problems" begin in elementary school with Johnny who has three apples and eats one and end in calculus with related rate problems.

The problems ask us to determine quantities and/or rates of change in quantities in situations involving several variables. These are called ***related rate problems*** because as one quantity changes it causes a change in another quantity. Think of a circle. If the radius increases or decreases at a particular rate, it will effect the rate at which the area of the circle increases or decreases…they are ***related rates***.

We already know everything we need to solve the problems. Like all word problems, the main task is in setting up the solution. Here are some general guidelines we will follow:

I. Draw a picture when possible. Label all elements of the diagram. If a particular distance is changing, label it with a variable—even if a numerical value is given. Avoid putting numbers on any distance that is changing.

II. Write down all the given information. If we've labeled a particular segment x and the problem tells us it is increasing at 5 feet per minute, we are being told that $\frac{dx}{dt} = 5$ where $\frac{dx}{dt}$ represents the rate of change in x with respect to time.

III. Write down the problem statement. Generally it will sound something like: Find $\frac{dr}{dt}$ when $r = 3$ and $\frac{dA}{dt} = 2$.

IV. Write an equation which ties together all the variables in the problem. This equation will normally be the Pythagorean theorem, a known volume or area formula, or a distance equation of some type.

V. Implicitly differentiate your equation with respect to t (time). Do not plug in any given numbers until after you have differentiated!

VI. Substitute any given information and solve for the required quantity or rate of change.

Categories of related rate problems

Although there are an infinite variety of related rate problems, there are several types which can be found in every calculus textbook published since L'Hopital put his text on the bookshelf. A short list is given below with a suggestion for setting up an equation to solve the problem.

The Ladder Problem: Most often employs the Pythagorean theorem. (Likely to be voted most famous related rate problem type!)

The Shadow Problem: Use similar triangles. The problem most often involves a person walking away from a suspended light.

The Kite Problem: Susie flies a kite and lets out the string so that the kite moves away at a constant altitude. Use the Pythagorean theorem.

The Intersection Problem: Cars, boats, trains approaching an intersection. Use the Pythagorean theorem or, on occasion, similar triangles.

The Sphere Problem: Snowballs, tumors or melting ice on a semicircular dome. Use the volume of a sphere.

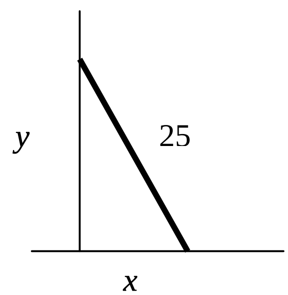
The Cone Problem: Voted most difficult by students worldwide. Usually involves liquid of some type being pumped into a tank in the form of a inverted right circular cone. Use the formula for the volume of a cone. The formula involves both an r (radius) and an h (height). In most problems you will have to set up similar triangles to eliminate the r variable.

The Searchlight Problem: A searchlight on an airplane or sweeping along a shoreline. These problems generally involve setting up a right triangle and then using either sine or tangent.

The only way to learn and become proficient at related rate problems is to do lots of them.

Example 1

A ladder 25 feet long is leaning against a vertical wall. If the bottom of the ladder is pulled away from the wall at a constant rate of 3 feet per second, how fast is the top of the ladder sliding down the wall when the bottom is 15 feet from the wall?



The bottom of the ladder is being pulled away from the wall at 3 feet per second. This means that the distance x is changing at 3 feet per second therefore $\frac{dx}{dt} = 3$ (it's positive because x is getting larger).

We need to find $\frac{dy}{dt}$ when $x = 15$.

$$x^2 + y^2 = 25^2$$
$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

When $x = 15$, $y = 20$ and we know $\frac{dx}{dt} = 3$. Substituting yields

$$2(15)(3) + 2(20)\tfrac{dy}{dt} = 0$$

$$\frac{dy}{dt} = -\frac{45}{20}$$

$$\frac{dy}{dt} = -\frac{9}{5}$$

Therefore, the top of the ladder is moving down the wall at 9/5 feet per second when the bottom is 15 feet from the wall.

Example 2

A spherical balloon is being inflated so that its volume is increasing at the rate of 5 cubic meters per minute. At what rate is the diameter increasing when the diameter is 12 meters?

The volume of a sphere is given by $V = \frac{4}{3} p r^3$. Since the problem is stated in terms of diameter $\left(r = \frac{d}{2}\right)$,

this formula can be changed to $V = \frac{p}{6} d^3$

We are given that $\frac{dV}{dt} = 5$.

We need to find $\frac{dd}{dt}$ when $d = 12$.

$$V = \tfrac{p}{6} d^3$$

$$\tfrac{dV}{dt} = \tfrac{p}{2} d^2 \tfrac{dd}{dt}$$

Substituting $d = 12$ and $\frac{dV}{dt} = 5$ we obtain

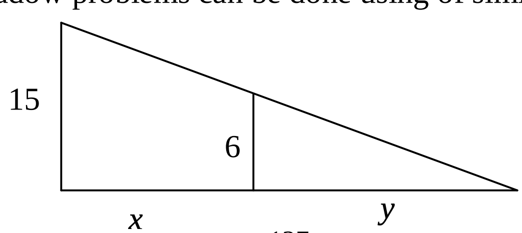
$$\frac{dd}{dt} = \frac{5}{72p}$$

Thus, the diameter is increasing at $\frac{5}{72p}$ (≈ 0.022) meters per minute.

Example 3

A light is hung 15 feet above a straight horizontal path. If a man 6 feet tall is walking away from the light at a rate of 5 feet per second, how fast is his shadow lengthening when he is 20 feet from the wall? How fast is the tip of his shadow moving?

Most shadow problems can be done using of similar triangles.



In our diagram, x is the horizontal distance from the man to a point directly under the light. y is the length of his shadow. We are given that $\frac{dx}{dt}=5$. We need to find $\frac{dy}{dt}$ when $x=20$.

The large triangle with 15 as its vertical side and the small triangle with 6 as its vertical side are similar triangles and so the following relationship can be written:

$$\frac{15}{x+y}=\frac{6}{y}$$

Cross multiplying and then simplifying yields,

$$15y=6x+6y$$

$$9y=6x$$

$$3y=2x$$

Differentiating with respect to t,

$$3\frac{dy}{dt}=2\frac{dx}{dt}$$

$$\frac{dy}{dt}=\frac{3}{2}\frac{dx}{dt}$$

For $\frac{dx}{dt}=5$ we obtain,

$$\frac{dy}{dt}=\frac{15}{2}$$

Therefore, the shadow is lengthening at 15/2 feet per second.

Now, the tip of the shadow is moving under two influences—the shadow itself is getting longer and the man is moving, therefore the tip of the shadow is moving at

$$\frac{dx}{dt}+\frac{dy}{dt}=5+\frac{15}{2}$$

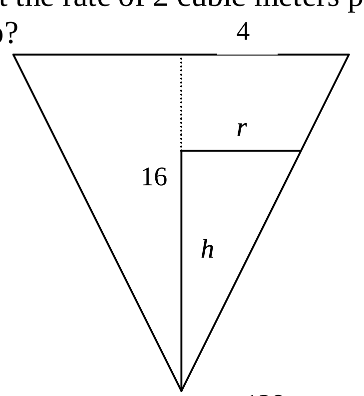
$$=\frac{25}{2}$$

Thus, the tip of the shadow is moving at 25/2 feet per second.

(Note that in this problem, the rate of change in the length of the shadow is independent of how far the person is from the light.)

Example 4

A tank is in the shape of an inverted cone having an altitude of 16 meters and a radius of 4 meters. Water is flowing into the tank at the rate of 2 cubic meters per minute. How fast is the water level rising when the water is 5 meters deep?



We will label the height of the water h and the radius of the water r.

The r can be expressed in terms of h by using the proportion

$$\frac{16}{h}=\frac{4}{r}\Rightarrow r=\frac{h}{4}$$

The volume of a cone of this type is $V = \frac{1}{3} p r^2 h$. For $r = \dfrac{h}{4}$ the formula becomes $V = \frac{p}{48} h^3$.

We now differentiate with respect to *t*.

$$V = \frac{p}{48} h^3$$

$$\frac{dV}{dt} = \frac{p}{16} h^2 \frac{dh}{dt}$$

For $h = 5$ and $\frac{dV}{dt} = 2$

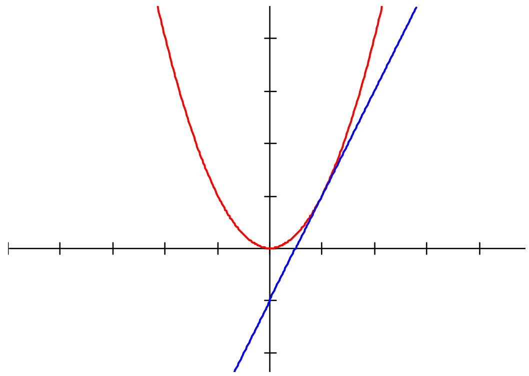
$$\frac{dh}{dt} = \frac{32}{25p}$$

Therefore, the height of the water is increasing at $\frac{32}{25p}$ meters per minute.

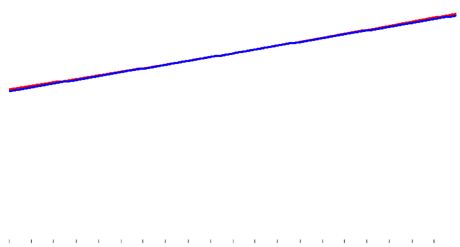
Local Linearity and Linearizations

Local linearizations

Take a look at the graph of $f(x) = x^2$ and the tangent to it at $x = 1$.



Now, let's zoom in very close to the point (1, 1). We will use $x_{\min}$ = .9 and $x_{\max}$ = 1.1



Notice how little difference there is between $f(x) = x^2$ *and* $y = 2x - 1$ close to x = 1! We could get very good estimates of function values for f near $x = 1$ by finding the value of $2x - 1$. Take a look at the chart below.

x	$f(x)$	$2x - 1$
1.1	1.210	1.200
1.01	1.020	1.020
1.001	1.002	1.002
.9	.810	.800
.95	.903	.900
.99	.980	.980

Note that as we get closer and closer to x = 1 from either direction, values for $f(x)$ and $2x - 1$ get very close.

This illustrates the basic idea of local linearity…that all differentiable functions are locally linear and thus can be approximated by using a linear function.

Now, you may have noticed that all we did to linearize $f(x)=x^2$ *at* $x=1$ was to write an equation of a tangent to f at $x=1$. You're right…that's all we did and that's all a linearization of a function is.

Now, we normally denote a linearization (as opposed to a tangent) as $L(x)$.

Consider the point-slope form of a line

$$y-y_1=m(x-x_1)$$

Solving for y gives

$$y=y_1+m(x-x_1)$$

At $x=a$, $m=f'(a)$ and $y_1=f(a)$ so

$$y=f(a)+f'(a)(x-a)$$

or

$$L(x)=f(a)+f'(a)(x-a)$$

In general, the linearization of f at $x=a$ is written

$$L(x)=f(a)+f'(a)(x-a)$$

Example 1

Linearize $f(x)=x^3-x$ at $x=2$

Since $f(2)=6$ and $f'(x)=3x^2-1 \rightarrow f'(2)=11$ we get

$$L(x)=6+11(x-2)$$

or

$$L(x)=11x-16$$

Example 2

Use an appropriate linearization to approximate $\sqrt{49.1}$ and then use your calculator to verify your result.

$$\text{Let } f(x)=\sqrt{x} \text{ and linearize } f \text{ at } x=49$$

$$\text{Since } f(49)=7 \textit{ and } f'(x)=\frac{1}{2\sqrt{x}} \rightarrow f'(49)=\frac{1}{14} \text{ we get}$$

$$L(x)=7+\tfrac{1}{14}(x-49)$$

Now, to find $\sqrt{49.1}$ we just find $L(49.1)$

$$\begin{aligned} L(49.1) &= 7+\tfrac{1}{14}(49.1-49) \\ &= 7+\left(\tfrac{1}{14}\right)\left(\tfrac{1}{10}\right) \\ &= 7+\tfrac{1}{140} \\ &\approx 7.007 \end{aligned}$$

Notice that as x goes from 49 to 49.1, the square root goes from 7 to $7+\frac{1}{140}$

Note: A linearization is actually something called a *first order Taylor Polynomial*. Any differentiable function can be approximated by a polynomial to whatever degree of accuracy we want.

Error Analysis in Linearizations

There are times when we will want to know the interval over which our linearization is accurate to within a specified range. The question is usually phrased in these terms: "For what values of x is the linearization of f accurate to within .5 of the actual function value.

If the allowable error is denoted E, and if we want to know the interval over which our linearization will be within E units of the actual function value, we want to know the values of x for which the following inequality will be true.

$$f(x)-E<L(x)<f(x)+E$$

If $f(x)-E$, $f(x)+E$ and $L(x)$ are all graphed on the same axis, we would want to know when our linearization is above *L(x) - E* and below *L(x) + E*. Depending on the function, the linearization and *E*, many different situations could arise.

The linearization could

- intersect each of the shifted functions once…resulting in an interval (x_1, x_2).
- the linearization could be above the downward shifted function for all x and intersect the upward shifted function twice…resulting in an interval (x_1, x_2).
- the linearization could be below the upward shifted function for all x and intersect the downward shifted function twice…resulting in an interval (x_1, x_2).
- the linearization could be above the downward shifted function for all x and only intersect the upward shifted function once…in which case you would get an interval which looked like $(-\infty, x_1)$ or (x_1, ∞).
- the linearization could be below the upward shifted function for all x and only intersect the downward shifted function once…in which case you would get an interval which looked like $(-\infty, x_1)$ or (x_1, ∞).

Just graph the linearization and the shifted functions and find the interval which satisfies the inequality $f(x)-E<L(x)<f(x)+E$.

Example 3

For what values of x will the linearization $\frac{1}{\sqrt{4-x}} \approx \frac{1}{2}+\frac{x}{16}$ be accurate to within .1?

This question tells us that the linearization of $f(x)=\frac{1}{\sqrt{4-x}}$ at some value of x (not given) is $L(x)=\frac{1}{2}+\frac{x}{16}$. They want to know for what values of x is will this linearization yield function values that are within .1 of the actual function values. A portion of our work has been done…they gave us the linearization!

We need to solve the inequality

$$\frac{1}{\sqrt{4-x}}-.1<\frac{x+8}{16}<\frac{1}{\sqrt{4-x}}+.1$$

Graphing these three expressions, we see that $\frac{x+8}{16}<\frac{1}{\sqrt{4-x}}+.1 \;\forall x$ so all we need to do is find the solution (via calculator!) of $\frac{1}{\sqrt{4-x}}-.1<\frac{x+8}{16}$.

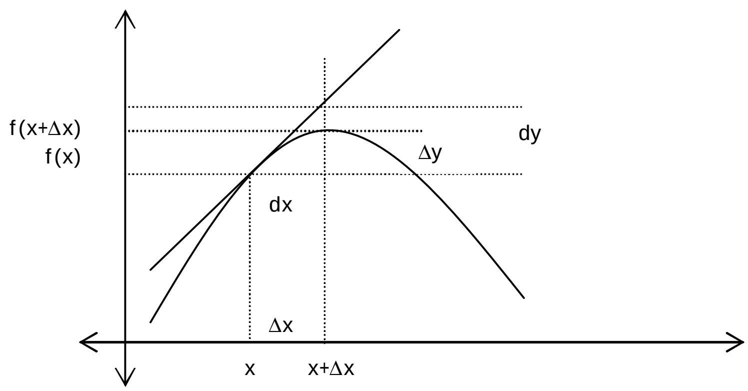
Since $\frac{1}{\sqrt{4-x}}-.1<\frac{x+8}{16}$ when x$\in$ (–3.912, 2.144) we can say that the linearization $\frac{1}{\sqrt{4-x}} \approx \frac{1}{2}+\frac{x}{16}$ will be accurate to within .1 when x$\in$ (–3.912, 2.144).

Differentials

Introduction

Differentials are very closely related to linearizations—they are another natural consequence of local linearity. Many problems that can be done with linearizations (like estimating the value of a square root) can be done using differentials.

Up until now, we have used $\frac{dy}{dx}$ as a symbol for the derivative of y with respect to x. In fact, $\frac{dy}{dx}$ is a fraction. The dy and the *dx*, as separate entities are called ***differentials***. *dx* is called the differential of *x* and *dy* the differential of *y*. To see this, take a look at the diagram below.



We know that the slope of the tangent to the curve at x is given by the derivative, $\frac{dy}{dx}$. We also know that the slope of a line is given by $\frac{rise}{run}$. The tangent line "runs" dx and "rises" dy…thus *dy* and *dx* represent real numbers!

Definitions

It's time for a few definitions. First of all we know that $\frac{dy}{dx} = f'(x)$. Since the dy and dx are differentials, we can multiply both sides by dx and get:

$$dy = f'(x)\,dx$$

From the diagram we can see that:

$$\Delta y = f(x+\Delta x) - f(x)$$

Δy is the **exact change** in function values as x goes from x to $x + \Delta x$.

Notice that as Δx gets smaller, the difference between Δy and dy becomes smaller and smaller.

This is why we can say that dy is the **approximate change** in function values as x goes from x to $x+\Delta x$.

$$dy \approx \Delta y \text{ for small } \Delta x$$

Finally, from the diagram we see that:

$$dx = \Delta x$$

Example 1

If $y = x^3 + 2x^2$, for $x = 2$ and $dx = .1$, find Δy and dy.

$$\frac{dy}{dx} = 3x^2 + 4x$$
$$dy = (3x^2 + 4x)\,dx$$
For $x = 2$ and $dx = .1$ we get

$$dy = (20)(.1) = 2$$

This means that as x goes from 2 to 2.1, the function value changes by approximately 2.

Now, $\Delta y = f(x+\Delta x) - f(x)$ so for $x = 2$ and $x + \Delta x = 2.1$

$$\Delta y = f(2.1) - f(2)$$
$$\Delta y = 2.081$$

This means that as x goes from 2 to 2.1, the function value actually changes by 2.081.

Notice that $\Delta y - dy \approx \frac{81}{1000}$ so our approximation was pretty good!

Example 2

Find the volume of glass needed to make a hollow sphere whose inner radius is 2 inches and is to be .01 inches thick.

Of course we could just calculate two exact volumes and subtract…

$$\Delta V = V(r+\Delta r) - V(r)$$
$$\Delta V = \frac{4}{3}p\,(2.01)^3 - \frac{4}{3}p\,(2.01)^3$$
$$\Delta V = .505$$

Thus the volume of the glass would be .515 cubic inches.
It is quicker to just find the change in the volume as the radius changes from 2 to 2.01 inches.

$$V = \frac{4}{3} p r^3$$

$$\frac{dV}{dr} = 4p r^2 dr$$

$$dV = 4p r^2 dr$$

For $r = 2$ and $dr = .01$

$$dV \approx .503$$

Thus the approximate change in volume is .503 cubic inches—only 1/500th of a cubic inch from the "exact" value.

Example 3

Use differentials to approximate the value of $\sqrt{81.7}$.

Let $y = \sqrt{x}$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

$$dy = \frac{1}{2\sqrt{x}} dx$$

For $x = 81$ and $dx = .7$

$$dy \approx .039$$

This is the change in function values as x goes from 81 to 81.7

$$\therefore \sqrt{81.7} \approx 9 + .039 \approx 9.039$$

Linearizations and differentials

The value of differentials comes from the idea that many times it is easier (or all that is necessary) to calculate an approximate change in function values rather than calculating an exact change. We saw a similar situation with linearizations…it was sometimes easier (or all that was necessary) to approximate a function value than finding an exact value.

Note that both linearizations and differentials are based on a tangent to a curve. Many problems that we have done with linearizations, we could have done with differentials. Consider the following example.

Example 4

Use a linearization of $f(x)=\sqrt{x}$ at $x=25$ to approximate $\sqrt{26}$. Then use differentials to estimate $\sqrt{26}$.

The linearization of $f(x)=\sqrt{x}$ at $x=25$ is

$$L(x)=5+\frac{1}{10}(x-25)$$

$$\text{Now } L(26)=5+\frac{1}{10}(26-25)$$

$$L(26)=5+\frac{1}{10}$$

$$\therefore \sqrt{26}\approx 5.1$$

To use differentials,

$$\text{Let } y=\sqrt{x}$$

$$\frac{dy}{dx}=\frac{1}{2\sqrt{x}}$$

$$dy=\frac{1}{2\sqrt{x}}dx$$

For $x=25$ and $dx=1$

$$dy=\frac{1}{10}(1)$$

$$dy=.1$$

This means that as x goes from 25 to 26, the square root increases by .1

$$\therefore \sqrt{26}\approx 5+.1$$

$$\therefore \sqrt{26}\approx 5.1$$

The decision to use a linearization or differentials depends on the problem at hand. Most problems that can be done with one technique can also be done with the other.

L'Hopital's Rule

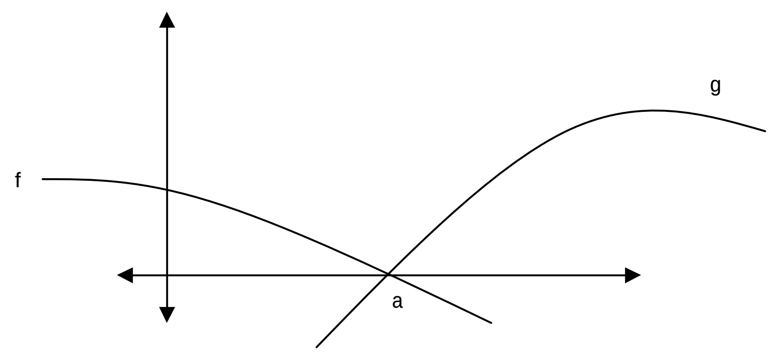
Introduction

L'Hopital's rule is a theorem used to find limits when the result of "plugging in a" is indeterminant. Indeterminant forms include $\frac{0}{0}$ and $\frac{\infty}{\infty}$. We have chosen to introduce L'Hopital's Rule now because local linearity can be used to derive it. Once we have L'Hopital's Rule, we will no longer have to factor or rationalize to find limits where we get zero over zero or infinity over infinity.

Historical note: Although the theorem is called "L'Hopital's Rule, the Marquis de L'Hopital (1661-1704) did not actually discover it. It was discovered by a member of the famous Bernoulli family, John Bernoulli (1667-1748). At the time of its discovery, L'Hopital was Bernoulli's patron. In exchange for L'Hopital's sponsorship, Bernoulli (an up and coming mathematician…without much money) agreed to show all results to L'Hopital first. L'Hopital published one of the very first Calculus texts and in it included Bernoulli's new rule. Although L'Hopital did not actually take credit for it, because it appeared in his text, and the text became popular, the rule came to be known as L'Hopital's Rule instead of Bernoulli's rule.

Derivation

If $\lim_{x\to a}\frac{f(x)}{g(x)}$ is of indeterminant form, it means that as $\lim_{x\to a} f(x)=0$ *and* $\lim_{x\to a} g(x)=0$. A visualization can be seen in the graph below.



Now, let's linearize both f and g at $x=a$.

The linearization of *f* at $x=a$ is $L_f(x)=f(a)+f'(a)(x-a)$ and the linearization of *g* at $x=a$ is $L_g(x)=g(a)+g'(a)(x-a)$. We now replace *f* and *g* in the original limit with their respective linearizations. This is allowed because we are taking a limit as $x\to a$ and the closer we get to *a* the closer the linearizations are to the actual functions.

We now have

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f(a) + f'(a)(x-a)}{g(a) + g'(a)(x-a)}$$

We also already know that as $x \to a$, both $f(x) \to 0$ and $g(x) \to 0$. This results in

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(a)(x-a)}{g'(a)(x-a)}$$

The common factor $(x-a)$ in the numerator and demoninator can now be reduced (because we are never actually letting $x = a$)

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(a)}{g'(a)}$$

Because $x \to a$ the statement above can be written as

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

This startling result is known as L'Hopital's rule.

If $\lim_{x \to a} \frac{f(x)}{g(x)}$ is of indeterminant form, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$.

Simply (and always dangerously) stated, L'Hopital's rule says that if the limit of a quotient appears to be of indeterminant form, the the limit of the quotient of two functions is the same as the limit of the derivatives of the two functions.

L'Hopital's Rule is especially useful when trying to find limits of quotients that involve trigonometric functions.

Example 1

Find $\lim_{x \to 5} \frac{x^2 - 25}{x - 5}$.

If we "plug in" the 5, we get $\frac{0}{0}$ so

$$\lim_{x \to 5} \frac{x^2 - 25}{x - 5} = \lim_{x \to 5} \frac{2x}{1} = 10$$

Example 2

Find $\lim_{x \to 9} \frac{\sqrt{x}-3}{x-9}$

If we "plug in" the 9, we get $\frac{0}{0}$,an indeterminant form so

$$\lim_{x \to 9} \frac{\sqrt{x}-3}{x-9} = \lim_{x \to 9} \frac{\frac{1}{2\sqrt{x}}}{1} = \frac{1}{6}$$

Example 3

Find $\lim_{x \to 0} \frac{\sin x}{x}$ (we already know this as a theorem, but watch how this works out!)

If we "plug in" the 0 we get $\frac{0}{0}$ so, applying L'Hopital's rule

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = 1$$

Newton's Method

Introduction

Newton's Method is a method used to approximate the zeros of functions. Again, it is a natural consequence of local linearity. Of course, with today's computers and calculators, there is little need to estimate zeros using this technique by hand but many computers and calculators actually use this algorithm to find approximate zeros.

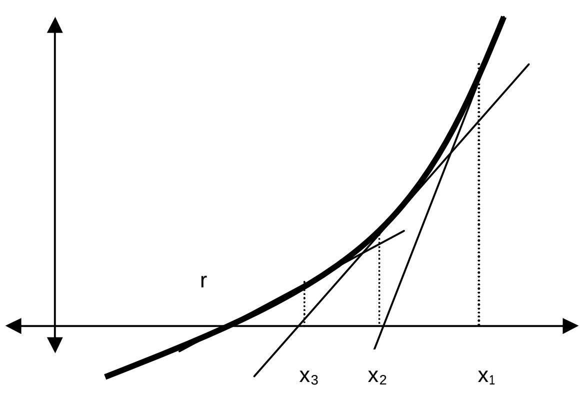
The Process

Newton's Method is an example of an iterative process. An iterative process uses the result from a calculation as input. For example if we iterated x^2 starting at x = 2 we would get: 2, 16, 256, 65536, … The first result was $2^2 = 4$, the 4 was now used as input to get 4^2 =16, etc.

Newton's Method goes like this:

- An initial approximation is made for the value of the root, called x_1.
- The function is linearized at x_1.
- The x-intercept of this linearization is caluclated.
- This x-intercept becomes the next approximation, x_2, of the zero.
- The function is linearized at x_2 and the process is repeated until the desired level of accuracy is obtained.

This process can be seen in the diagram below.



Derivation

If we obtained a tangent to *f* at x_1 (linearized *f* at x_1) its slope would be $f'(x_1)$.

The equation of the first tangent is

$$y - f(x_1) = f'(x_1)(x - x_1)$$

The x-intercept of this line is $(x_2, 0)$. The x-coordinate is called x_2 because it will become our second estimate of the root. Now, letting $y = 0$ and $x = x_2$.

$$0 - f(x_1) = f'(x_1)(x_2 - x_1)$$

Solving for x_2

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

The next approximation follows the same algorithm so the *n*th approximation becomes

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Newton's Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Newton's Method problems are calculator driven. To perform Newton's Method on our calculator, follow these steps:

- Set the desired number of decimal places in the MODE
- Enter the function in y1
- In y2, enter the derivative—y2=d(y1(x),x)
- Return to the HOME screen
- In the entry line enter x-(y1(x)/y2(x))|x=x_1 (Here the x_1 is your initial guess.)
- Instead of pressing ENTER, use GREEN ENTER to get a decimal approximation
- Erase the x_1 you used and replace it with Ans(1)
- Now, if you continue to press GREEN ENTER you will get successive approximations! The calculator is using the anwer from the previous calculation to perform the next.
- Continue until you get the same approximation twice

Example 1

Find the real root of $x^3 - 4x^2 - 2 = 0$ to four decimal places. Use $x_1 = 4.5$

Set the MODE to FIX 4
Enter: y1=$x^3 - 4x^2 - 2$
Enter: y2=d(y1(x),x)
Return to HOME
Enter: x-(y1(x)/y2(x))|x=4.5
Press GREEN ENTER

The result is now x_2, our second approximation. $x_2 = 4.1717$

Backspace over our 4.5 and press 2nd ENTER to get x = ans(1)

Press GREEN ENTER

The result is now x3, our third approximation. $x_3 = 4.1192$

We already have ans(1) on the entry line so just press GREEN ENTER again.

The result is now x_4, our fourth approximation. $x_4 = 4.1179$

Press GREEN ENTER.

The result is now x_5, our fifth approximation. $x_5 = 4.1179$

Since 4.1179 now repeats itself, the approximation of the real zero to four decimal places is $x \approx 4.1179$.

Example 2

Find the value of $\sqrt{3}$ to five decimal places.

To do this, find the solution to the equation $x^2 - 3 = 0$, since the solution will be $\sqrt{3}$.

Just repeat Newton's Method.

Example 3

Find the intersection of $y = \frac{1}{2}x$ and $y = \sin x$ to five decimal places.

To find the intersection, set the two equations equal to each other

$$\frac{1}{2}x = \sin x$$

Get everything to one side (Newton's Method is used to approximate zeros!)

$$\frac{1}{2}x - \sin x = 0$$

Now use Newton's Method. Make sure your calculator is set to FIX 5.

Problems with Newton's Method

There are situation which may cause Newton's Method to go "nuts".

An inappropriate initial guess can lead to x-intercepts (next approximations) which move away from the desired root instead of getting closer to it. Try using Newton's Method to approximate the zero of $f(x) = \sqrt[3]{x}$ with an initial guess of $x_1 = .1$ The approximations do not converge on the root, instead they move continually away from it!

On occasion, you can get bizzarre results. Try approximation the zero of $f(x) = x^3 - 10x^2 + 22x + 6$ with $x_1 = 2$. You will get approximations that alternate between 2 and 5!

In the end, if the method is not giving you the result you need, you may need to change your initial guess…that usually solves the problem.