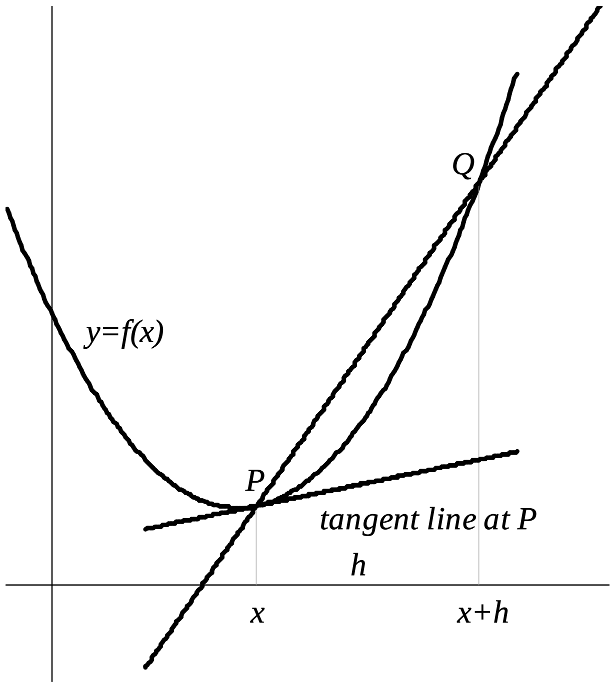
The Derivative

Introduction

In a previous section, we estimated the slope of a tangent to a curve at some point *P*. We accomplished this by considering a point *Q* a short distance from *P*. We calculated the slope of the secant line passing through *P* and *Q* as the point *Q* gets closer and closer to *P*. The closer *Q* gets to *P*, the closer the slope of the secant is to the slope of the tangent at *P*. In this section, we will learn how to go beyond estimating and actually find the exact slope of a tangent to a curve at a point.

The definition of derivative

Consider the diagram below.



The slope of the secant line through *P* and *Q* is given by $\frac{f(x+h)-f(x)}{h}$. As point Q moves closer to point *P*, the secant line and the tangent line at *P* become closer to being the same line. All we need is a process that allows us to let the distance *h* get arbitrarily small. That process, of course, is limits! If we take the limit of $\frac{f(x+h)-f(x)}{h}$ as $h \to 0$, the slope of the secant and the slope of the tangent will be exactly the same. This limit, $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$, is called the ***derivative*** of *f* at *x* and is denoted $f'(x)$.

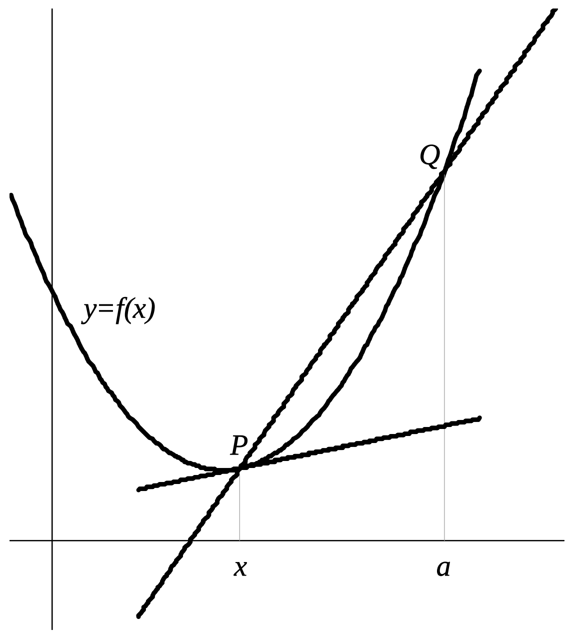
Definition of derivative

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

In some instances, instead of using *h* to denote a small change in *x*, the symbol Δx is used. If we use Δx, the definition of the derivative becomes $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$.

The quotient we started with, $\frac{f(x+h) - f(x)}{h}$, is called a ***difference quotient.*** A difference quotient tells us the <u>average</u> rate of change in function values as *x* goes from *x* to $x+h$. The limit of this difference quotient as $h \to 0$ is called the derivative and tells us the <u>instantaneous</u> rate of change in function values at *x*.

There are many ways to set up the definition of the derivative. Consider the following diagram.



In this figure, instead of moving some distance *h* from *x* to get coordinates of point *Q*, we simply choose another *x*-coordinate, *a*. Now, the coordinates of point *Q* are $(a, f(a))$. The difference quotient which yields the slope of the secant through *P* and *Q* is given by $\frac{f(a) - f(x)}{a - x}$. In order to get the derivative, instead of letting a distance *h* go to zero, we let *a* approach *x*. The definition of the derivative now becomes $f'(x) = \lim_{a \to x} \frac{f(a) - f(x)}{a - x}$. The derivative is simply a limit of a difference quotient. Listed below are several ways to express the derivative.

Definitions of Derivative

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$f'(m) = \lim_{n \to m} \frac{f(n) - f(m)}{n - m}$$

$$f'(b) = \lim_{a \to b} \frac{f(a) - f(b)}{a - b}$$

$$f'(x) = \lim_{a \to 0} \frac{f(x+a) - f(x)}{a}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Many students get into a lazy habit of saying "…the derivative is the slope of a tangent…" or just "…the derivative is the slope…". Actually, the derivative is a function or expression which can yield the slope of a tangent at any point.

How we interpret the derivative depends on the problem. In its purest sense the derivative is a function which yields the instantaneous rate of change in function values at any point $(x, f(x))$. If we begin with a position function, the derivative is a function which will yield the instantaneous velocity at any time *t*. If we begin with a function that describes the temperature of a pond over time, the derivative is a function which will yield the instantaneous temperature at any time *t*.

Consider a function *V(r)* which describes the rate of change of the volume of an object with a given radius. The difference quotient $\frac{V(r_2) - V(r_1)}{r_2 - r_1}$ tells us the average rate of change in the volume of the object as the radius changes from r_2 to r_1. If we take the limit of this difference quotient as $r_2 \to r_1$, we would obtain $V'(r_1)$, which will yield the instantaneous rate of change in the volume at the instant when the radius is r_1. The difference $r_2 - r_1$ could also be written Δr and so the derivative could be expressed as $V'(r_1) = \lim_{\Delta r \to 0} \frac{V(r_2) - V(r_1)}{\Delta r}$.

Now that we have the definition of derivative, we can move on to actually finding them. In later sections, we will learn many theorems that will allow us to find derivatives quickly and easily. For now, we will use the definition of the derivative to find derivatives.

Example 1

Given $f(x) = 6x - 7$, find $f'(x)$.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{[6(x+h) - 7] - [6x - 7]}{h} \\ &= \lim_{h \to 0} \frac{[6x + 6h - 7] - [6x - 7]}{h} \\ &= \lim_{h \to 0} \frac{6x + 6h - 7 - 6x + 7}{h} \\ &= \lim_{h \to 0} \frac{6h}{h} \\ &= \lim_{h \to 0} 6 \\ &= 6 \end{aligned}$$

$$\therefore f'(x) = 6$$

Example 2

Given $f(x) = x^2 + 5x$, find $f'(x)$.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{[(x+h)^2 + 5(x+h)] - [x^2 + 5x]}{h} \\ &= \lim_{h \to 0} \frac{[x^2 + 2xh + h^2 + 5x + 5h] - [x^2 + 5x]}{h} \\ &= \lim_{h \to 0} \frac{2xh + h^2 + 5h}{h} \\ &= \lim_{h \to 0} \frac{h(2x + h + 5)}{h} \\ &= \lim_{h \to 0} [2x + h + 5] \\ &= 2x + 5 \end{aligned}$$

$$\therefore f'(x) = 2x + 5$$

Example 3

Given $f(x)=\dfrac{3x+2}{2x-5}$ find $f'(x)$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{\dfrac{3(x+h)+2}{2(x+h)-5}-\dfrac{3x+2}{2x-5}}{h} \\ &= \lim_{h\to 0}\frac{\dfrac{3x+3h+2}{2x+2h-5}-\dfrac{3x+2}{2x-5}}{h} \end{aligned}$$

To simplify our work we will find a common denominator in the numerator and factor the h out in front as $\dfrac{1}{h}$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\left[\frac{1}{h}\left(\frac{-19h}{(2x+2h+5)(2x-5)}\right)\right] \\ &= \lim_{h\to 0}\frac{-19}{(2x+2h+5)(2x-5)} \\ &= \frac{-19}{(2x+5)(2x-5)} \\ &= \frac{-19}{(2x-5)^2} \end{aligned}$$

Example 4

Given $f(x)=\sqrt{x+3}$, find $f'(x)$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{\sqrt{(x+h)+3}-\sqrt{x+3}}{h} \end{aligned}$$

To simplify, we will rationalize the numerator. We will not simplify any multiplications in the resulting denominator.

$$\begin{aligned} &= \lim_{h\to 0}\frac{\sqrt{(x+h)+3}-\sqrt{x+3}}{h}\bullet\frac{\sqrt{(x+h)+3}+\sqrt{x+3}}{\sqrt{(x+h)+3}+\sqrt{x+3}} \\ &= \lim_{h\to 0}\frac{(x+h+3)-(x+3)}{h\left(\sqrt{(x+h)+3}+\sqrt{x+3}\right)} \end{aligned}$$

$$= \lim_{h\to 0} \frac{h}{h\left(\sqrt{(x+h)+3}+\sqrt{x+3}\right)}$$

$$= \lim_{h\to 0} \frac{1}{\left(\sqrt{x+h+3}+\sqrt{x+3}\right)}$$

$$= \frac{1}{\left(\sqrt{x+3}+\sqrt{x+3}\right)}$$

$$= \frac{1}{2\sqrt{x+3}}$$

$$\therefore\ f'(x) = \frac{1}{2\sqrt{x+3}}$$

Example 5

Find an equation of the tangent line to $f(x) = x^2 + x$ at $x = 2$.

To write an equation of a tangent line, all we need is a point and the slope.
The slope of the tangent can be obtained from the derivative.

Point

$$f(2) = 6$$
$$\therefore (2,6)$$

Slope

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{[(x+h)^2 + (x+h)] - [x^2 + x]}{h}$$

$$= \lim_{h\to 0} \frac{[(x^2 + 2xh + h^2 + x + h)] - [x^2 + x]}{h}$$

$$= \lim_{h\to 0} \frac{2xh + h^2 + h}{h}$$

$$= \lim_{h\to 0} \frac{h(2x + h + 1)}{h}$$

$$= \lim_{h\to 0} \left[2x + h + 1\right]$$

$$= 2x + 1$$

$$\therefore\ f'(x) = 2x + 1$$

Now, $f'(2) = 5$ $\therefore$ the slope of the tangent at the point (2,6) is 5.

Equation of the tangent

Using the point and slope we calculated
we obtain:

$$y - 6 = 5(x - 2)$$

Comments on notation

There are several ways that a derivative or the process of differentiation can be denoted. As we learn more about differentiation, we will learn more notation.

If we are given a function as in "$f(x) =$", the derivative would be denoted $f'(x)$. Of course, if we are given "$Q(x) =$", the derivative would be denoted $Q'(x)$.

If we are given an equation of the form "$y =$", the derivative is denoted $\frac{dy}{dx}$. For now, $\frac{dy}{dx}$ is only a symbol, not a fraction. Later on we'll see that it actually is a quotient, but for now, it's just a symbol.

We will also see notation such as $D_x[x^2]$. This is something of a command which says, "Find the derivative of x^2." We would respond $D_x[x^2] = 2x$. The notation $\frac{d}{dx}\left(x^2\right)$ means exactly the same thing…find the derivative of x^2.

If we need to denote the value of the derivative of a function at a point, say $x = 7$, we would write $f'(7)$.

If we have a derivative such as $\frac{dy}{dx} = 3x - 1$, we could denote its value at $x = 7$ by writing $\left.\frac{dy}{dx}\right|_{x=7}$.

Differentiation Theorems

Introduction

We've been using the definition of derivative to find derivatives. While this is a perfectly acceptable method, it can get quite cumbersome…as we already know. In this section, we will prove several theorems related to differentiation which will make finding a derivative much quicker. Once we have these theorems, we will no longer have to rely on the definition of derivative to find derivatives. From now on, the only time we will use the definition of derivative to find a derivative is when the problem states "…using the definition of derivative, find…". Does this mean we are finished with the definition? No. Many of the theorems we are about to prove begin with the definition of derivative and we will return to the definition frequently throughout the course.

The derivative of a constant

Consider $f(x)=c$, where c is a constant.

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{c-c}{h} \\ &= 0 \end{aligned}$$

If $f(x)=c$ where c is a constant,

$$f'(x)=0$$

or

$$D_x[c]=0$$

Note that in the limit statement above, $c-c=0$ but h will never be zero. The fraction $\frac{c-c}{h}$ will always have a zero in the numerator and some arbitrarily small number in the denominator—thus the value of the fraction is zero.

Example 1

Given $f(x)=5$, find $f'(x)$.

$$f'(x)=0$$

The Power Rule

Before we move on to the proof, let's look for a particular pattern when binomials are expanded. Look at the following expansions:

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$
$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

Notice that the exponent on the binomial shows up as the coefficient of the second term in the expansion. For example, the first two terms of $(a+b)^6$ will be $a^6 + 6a^5b$. If we use this idea to expand $(x+h)^n$ we obtain an expression of the form $x^n + nx^{n-1}h + Bx^{n-2}h^2 + Cx^{n-3}h^3 + ... + h^n$ where *B*, *C* and so on represent the coefficients of each term after the second. (We can actually find these coefficients using the binomial theorem, but they are of no concern to us at this point!)

Now, consider $f(x) = x^n$ where *n* is any real number.

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h\to 0}\frac{(x+h)^n - (x^n)}{h}$$
$$= \lim_{h\to 0}\frac{(x^n + nx^{n-1}h + Bx^{n-2}h^2 + Cx^{n-3}h^3 + ...h^n) - (x^n)}{h}$$
$$= \lim_{h\to 0}\frac{x^n + nx^{n-1}h + Bx^{n-2}h^2 + Cx^{n-3}h^3 + ... + h^n - x^n}{h}$$
$$= \lim_{h\to 0}\frac{nx^{n-1}h + Bx^{n-2}h^2 + Cx^{n-3}h^3 + ... + h^n}{h}$$

Notice that the x^n term at the beginning and end of the numerator add to zero. Also notice that *h* is a factor of all the remaining terms of the numerator. We will factor out an *h* in the numerator and then reduce it with the *h* in the denominator.

$$= \lim_{h\to 0}\frac{h[nx^{n-1} + Bx^{n-2}h + Cx^{n-3}h^2 + ... + h^{n-1}]}{h}$$
$$= \lim_{h\to 0}\left[nx^{n-1} + Bx^{n-2}h + Cx^{n-3}h^2 + ... + h^{n-1}\right]$$

We now have an *h* as a factor of every term except the first. As $h \to 0$, all of these terms go to zero.

$$= nx^{n-1}$$

The Power Rule

If $f(x) = x^n$, where n is a real number,

$$f'(x) = nx^{n-1}$$

or

$$D_x[x^n] = nx^{n-1}$$

Example 2

Given $f(x) = x^9$, find $f'(x)$.

$$f'(x) = 9x^8$$

The derivative of $cf(x)$, sums and differences

We now introduce several important theorems. The proofs are very straightforward and are left for the reader.

$$D_x[cf(x)] = cf'(x)$$

This theorem states that the derivative of a function multiplied by a constant is equal to the constant multiplied by the derivative of the function. For example, if $f(x) = 4x^3$, the derivative becomes $f'(x) = 4[3x^2] = 12x^2$. Notice that a constant is handled differently depending on whether it is a "stand alone" constant $\left(D_x[c] = 0\right)$ or a constant that is a factor of a term.

$$D_x[f(x) + g(x)] = f'(x) + g'(x)$$

and

$$D_x[f(x) - g(x)] = f'(x) - g'(x)$$

These two theorems state that the derivative of a sum is the sum of the derivatives and the derivative of a difference is the difference of the derivatives.

The next example makes use of the theorems we have discussed thus far.

Example 3

Given $f(x) = 5x^4 - 3x^2 + 8$, find $f'(x)$.

$$f'(x) = 20x^3 - 6x$$

(Note that the derivative of the constant 8 was zero.)

Example 4

Given $f(x)=\sqrt{x}+\sqrt[5]{x^2}$ find $f'(x)$.

First, rewrite the function with rational exponents, then use the power rule.

$$f(x)=x^{1/2}+x^{2/5}$$
$$f'(x)=\tfrac{1}{2}x^{-1/2}+\tfrac{2}{5}x^{-3/5}$$
$$=\frac{1}{2\sqrt{x}}+\frac{2}{5\sqrt[5]{x^3}}$$

The Product Rule

We already know that the derivative of a sum is simply the sum of the derivatives, but is the derivative of a product the product of the derivatives? If it is, then the derivative of the function $f(x)=(x^3+x)(5x^2-2x)$ would be $f'(x)=(3x^2+1)(10x-2)$. Unfortunately, it's not that easy. The correct derivative is $f'(x)=(x^3+x)(10x-2)+(5x^2-2x)(3x^2+1)$. To differentiate a product we use something called "The Product Rule". Let's see where it comes from.

Let $p(x)=f(x)g(x)$. Using the definition of derivative we can write

$$p'(x)=\lim_{h\to 0}\frac{p(x+h)-p(x)}{h}.$$

Now, because $p(x)=f(x)g(x)$,

$$p'(x)=\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}$$

In order to move forward, we are going to subtract, and then add, the quantity $f(x+h)g(x)$ to the numerator. All we are actually doing is adding zero to the numerator in order to change the way it looks—a common tactic in proving theorems.

$$p'(x)=\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x+h)g(x)+f(x+h)g(x)-f(x)g(x)}{h}$$

We now separate the quotient into two parts.

$$p'(x)=\lim_{h\to 0}\left[\frac{f(x+h)g(x+h)-f(x+h)g(x)}{h}+\frac{f(x+h)g(x)-f(x)g(x)}{h}\right]$$

In the first fraction, $f(x+h)$ is a common factor and in the second fraction $g(x)$ is a common factor.

$$p'(x) = \lim_{h \to 0} \left[f(x+h) \frac{g(x+h)-g(x)}{h} + g(x) \frac{f(x+h)-f(x)}{h} \right]$$

Inside the brackets we now have four pieces and we will take the limit of each as $h \to 0$. Let's look at each one separately and then put them together.

$\lim_{h \to 0} f(x+h) = f(x)$. Here we simply let *h* go to zero.

$\lim_{h \to 0} \frac{g(x+h)-g(x)}{h} = g'(x)$. The limit statement is the definition of the derivative!

$\lim_{h \to 0} g(x) = g(x)$. This is true because $g(x)$ is not dependent on *h* at all…so it doesn't matter where *h* goes.

Finally, $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = f'(x)$. Again, the limit statement is the definition of derivative.

Let's put them together now. We'll begin with our last step in the proof.

$$p'(x) = \lim_{h \to 0} \left[f(x+h) \frac{g(x+h)-g(x)}{h} + g(x) \frac{f(x+h)-f(x)}{h} \right]$$

$$p'(x) = f(x)g'(x) + g(x)f'(x)$$

The Product Rule

If $h(x) = f(x)g(x)$ then

$$h'(x) = f(x)g'(x) + g(x)f'(x)$$

or

$$D_x[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

In words you will never forget, the product rule says,
"The derivative of a product is the first times the derivative of the second plus the second times the derivative of the first."

Note: Some students like to use the commutative property of addition to write the product rule as $D_x[f(x)g(x)] = f'(x)g(x) + g'(x)f(x)$. I suppose this is because some students are so anxious to take a derivative that they want to start right off with $f'(x)$. I HIGHLY DISCOURAGE THIS! It has been my experience that students who memorize the product rule "backwards" will more often than not memorize the next theorem (the quotient rule) incorrectly. I'm not sure why this happens, but it happens. Trust me. Please, when you recite the product rule, always begin with "…the **first** times the derivative of the second…"

We haven't yet run into any functions where the product rule is absolutely necessary. Consider the function $f(x) = (3x+2)(x-5)$. Since f is the product of two functions, we could use the product rule but is generally easier to simply multiply the two together and then use the power rule. We will make much more use of the product rule when we start running into more complicated functions such as $g(x) = x^2 \sin x$. To find the derivative of $g(x) = x^2 \sin x$, the product rule must be used because the product of x^2 and $\sin x$ cannot be simplified any further than it already is…unlike the product $(3x+2)(x-5)$, which can be simplified to $3x^2 - 13x - 10$.

Example 5

Given $h(x) = x^3 f(x)$, find $h'(x)$.

$$h'(x) = \left[x^3\right]\left[f'(x)\right] + \left[f(x)\right]\left[3x^2\right]$$

The Quotient Rule

The derivative of a product is not the product of the derivatives—we need the product rule to find the derivative of a product. Similarly, the derivative of a quotient requires a special rule…the quotient rule.

Consider the function $q(x) = \dfrac{f(x)}{g(x)}$. Using the definition of derivative we obtain,

$$\begin{aligned} q'(x) &= \lim_{h\to 0} \frac{q(x+h) - q(x)}{h} \\ &= \lim_{h\to 0} \frac{\dfrac{f(x+h)}{g(x+h)} - \dfrac{f(x)}{g(x)}}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left[\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}\right] \\ &= \lim_{h\to 0} \frac{1}{h}\left[\frac{f(x+h)g(x) - f(x)g(x+h)}{g(x)g(x+h)}\right] \end{aligned}$$

At this point we will subtract, then add $f(x)g(x)$.

$$q'(x) = \lim_{h\to 0} \frac{1}{h}\left[\frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{g(x)g(x+h)}\right]$$

We now, (1) distribute the $\frac{1}{h}$ through the numerator, (2) separate the numerator into two parts and (3) factor out common terms.

$$q'(x)=\lim_{h\to 0}\frac{g(x)\dfrac{f(x+h)-f(x)}{h}-f(x)\dfrac{g(x+h)-g(x)}{h}}{g(x)g(x+h)}$$

Taking the limit of each of the terms yields,

$$q'(x)=\frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}$$

The Quotient Rule

If $h(x)=\dfrac{f(x)}{g(x)}$, then

$$h'(x)=\frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}$$

or

$$D_x\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}$$

In words you will never forget, the quotient rule says,

"The derivative of a quotient is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator over the denominator squared."

Example 6

Given $f(x)=\dfrac{x-4}{8x+1}$, find $f'(x)$.

$$\begin{aligned} f'(x)&=\frac{(8x+1)(1)-(x-4)(8)}{(8x+1)^2}\\ &=\frac{8x+1-8x+32}{(8x+1)^2}\\ &=\frac{33}{(8x+1)^2}\end{aligned}$$

Example 7

Given $f(x)=\dfrac{10}{x^4}$, find $f'(x)$.

We will do this problem two ways—once with the quotient rule and again by rewriting the function to avoid using the quotient rule.

$$f'(x) = \frac{(x^4)(0) - (10)(4x^3)}{x^8}$$
$$= \frac{-40x^3}{x^8}$$
$$= -\frac{40}{x^5}$$

Now, rewrite the original function as $f(x) = 10x^{-4}$,

$$f'(x) = -40x^{-5}$$
$$= -\frac{40}{x^5}$$

Clearly, it is easier to differentiate after rewriting the function with negative exponents. When to rewrite a function and when not to is something that will become easier to determine as you find more and more derivatives.

Example 8

Given $y = \frac{\sqrt{x}-1}{\sqrt{x}+1}$, find $\frac{dy}{dx}$.

$$y = \frac{x^{1/2}-1}{x^{1/2}+1}$$
$$\frac{dy}{dx} = \frac{(x^{1/2}+1)\left(\frac{1}{2}x^{-1/2}\right) - (x^{1/2}-1)\left(\frac{1}{2}x^{-1/2}\right)}{(x^{1/2}+1)^2}$$
$$= \frac{1}{\sqrt{x}\left(\sqrt{x}+1\right)^2}$$

Example 9

Given $y = \frac{3}{x^3 - 6x + 8}$, find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{(x^3-6x+8)(0) - 3(3x^2-6)}{(x^3-6x+8)^2}$$
$$= \frac{18-9x^2}{(x^3-6x+8)^2}$$

Derivatives of the Trigonometric Functions

Introduction

We're going to get right to it in this section. Now that we have the definition of the derivative, the power rule, the product rule, the quotient rule and several other differentiation theorems, we're ready to find the derivatives of the six trigonometric functions. In order to avoid confusion with the hyperbolic trigonometric functions ($\sinh x, \cosh x, \text{etc.}$) we will use Δx instead of h in the definition of derivative.

Derivative of the sine function

Let $f(x) = \sin x$.

$$\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\sin(x+\Delta x) - \sin(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{[\sin x \cos \Delta x + \cos x \sin \Delta x] - \sin(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\sin x \cos \Delta x + \cos x \sin \Delta x - \sin x}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\sin x \cos \Delta x - \sin x + \cos x \sin \Delta x}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\sin x(\cos \Delta x - 1) + \cos x \sin \Delta x}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left[\frac{\sin x(\cos \Delta x - 1)}{\Delta x} + \frac{\cos x \sin \Delta x}{\Delta x}\right] \\
&= \lim_{\Delta x \to 0} \left[\sin x \frac{\cos \Delta x - 1}{\Delta x}\right] + \lim_{\Delta x \to 0} \left[\cos x \frac{\sin \Delta x}{\Delta x}\right]
\end{aligned}$$

From our previous work with limits of trigonometric functions we know that
$\lim_{\Delta x \to 0} \frac{\cos \Delta x - 1}{\Delta x} = 0$ and $\lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x} = 1$, therefore,

$$\begin{aligned}
f'(x) &= (\sin x)(0) + (\cos x)(1) \\
&= \cos x
\end{aligned}$$

If $f(x) = \sin x \to f'(x) = \cos x$

or

$D_x[\sin x] = \cos x$

Derivative of the cosine function

Let $f(x) = \cos x$.

$$\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\cos(x+\Delta x) - \cos(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{[\cos x \cos \Delta x - \sin x \sin \Delta x] - \cos(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\cos x \cos \Delta x - \sin x \sin \Delta x - \cos x}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\cos x \cos \Delta x - \cos x - \sin x \sin \Delta x}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left[\cos x \frac{\cos \Delta x - 1}{\Delta x} \right] - \lim_{\Delta x \to 0} \left[\sin x \frac{\sin \Delta x}{\Delta x} \right] \\
&= (\cos x)(0) - (\sin x)(1) \\
&= -\sin x
\end{aligned}$$

If $f(x) = \cos x \to f'(x) = -\sin x$

or

$D_x[\cos x] = -\sin x$

Derivative of the tangent function

We can combine the quotient rule with the derivatives of sine and cosine to find the derivative of the tangent function without resorting to the definition of derivative.

Let $f(x) = \tan x$.

$$\begin{aligned}
f(x) &= \tan x = \frac{\sin x}{\cos x} \\
f'(x) &= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} \\
&= \sec^2 x
\end{aligned}$$

If $f(x) = \tan x \to f'(x) = \sec^2 x$

or

$D_x[\tan x] = \sec^2 x$

Derivative of the secant function

Let $f(x) = \sec x$.

$$
\begin{aligned}
f(x) &= \sec x = \frac{1}{\cos x} \\
f'(x) &= \frac{(\cos x)(0) - (1)(-\sin x)}{\cos^2 x} \\
&= \frac{\sin x}{\cos^2 x} \\
&= \frac{1}{\cos x}\,\frac{\sin x}{\cos x} \\
&= \sec x \tan x
\end{aligned}
$$

If $f(x) = \sec x \rightarrow f'(x) = \sec x \tan x$

or

$D_x[\sec x] = \sec x \tan x$

Derivative of the cotangent function

Let $f(x) = \cot x$

$$
\begin{aligned}
f(x) &= \cot x = \frac{\cos x}{\sin x} \\
f'(x) &= \frac{(\sin x)(-\sin x) - (\cos x)(\cos x)}{\sin^2 x} \\
&= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} \\
&= -\frac{\sin^2 x + \cos^2 x}{\sin^2 x} \\
&= -\frac{1}{\sin^2 x} \\
&= -\csc^2 x
\end{aligned}
$$

If $f(x) = \cot x \rightarrow f'(x) = -\csc^2 x$

or

$D_x[\cot x] = -\csc^2 x$

Derivative of the cosecant function

Let $f(x) = \csc x$.

$$\begin{aligned} f(x) &= \csc x = \frac{1}{\sin x} \\ f'(x) &= \frac{(\sin x)(0) - (1)(\cos x)}{\sin^2 x} \\ &= \frac{-\cos x}{\sin^2 x} \\ &= -\frac{1}{\sin x}\,\frac{\cos x}{\sin x} \\ &= -\csc x \cot x \end{aligned}$$

If $f(x) = \csc x \rightarrow f'(x) = -\csc x \cot x$

or

$D_x[\csc x] = -\csc x \cot x$

Example 1

Given $f(x) = 3\sin x$, find $f'(x)$.

$$f'(x) = 3\cos x$$

Example 2

Given $f(x) = x^2 \sin x$, find $f'(x)$.

$$\begin{aligned} f'(x) &= (x^2)(\cos x) + (\sin x)(2x) \\ &= x^2 \cos x + 2x \sin x \end{aligned}$$

Example 3

Given $y = \dfrac{2\cos x}{x+1}$, find $\dfrac{dy}{dx}$.

$$\begin{aligned} \frac{dy}{dx} &= \frac{(x+1)(-2\sin x) - (2\cos x)(1)}{(x+1)^2} \\ &= \frac{-2(x+1)\sin x - 2\cos x}{(x+1)^2} \end{aligned}$$

Example 4

Given $f(t) = \sin t \tan t$, find $f'(t)$.

$$\begin{aligned} f'(t) &= (\sin t)(\sec^2 t) + (\tan t)(\cos t) \\ &= (\sin t)(\sec^2 t) + \frac{\sin t}{\cos t}\cos t \\ &= \sin t(\sec^2 t + 1) \end{aligned}$$

We should make note of the wonderful symmetry in the derivatives of the trigonometric functions. Sine becomes cosine, cosine becomes the negative of sine, etc. It's another reason the trigonometric functions are so important in mathematics and why mathematians love them so much. The trigonometric functions are extremely well-behaved.

Note: Notice that in all our current derivative theorems for the trigonometric functions, the argument of the trigonometric function is a single variable. Although we can find the derivative of $\sin p$, we cannot yet find the derivative of $\sin 7p$ or $\sin(9p+5)$. Functions and expressions such as these will require the chain rule…which is next up on our agenda!

The Chain Rule

Introduction

Although we have discussed quite a few theorems used for finding derivatives, we have confined ourselves to fairly simple functions and expressions. Right now we can, for example, find the derivative of $f(x) = x^7$, but not $f(x) = (4x-3)^7$ …at least not without expanding the binomial. We can find the derivative of $g(x) = \sin x$, but not $g(x) = \sin(x^3 - x)$ or $h(x) = \sec^8(5x-2)$. We can differentiate $y = \sqrt{x}$, but not $y = \sqrt{7x-3}$. Until now, we have never dealt with a composition of two or more functions. To differentiate a composition, we need the chain rule.

We use the product rule when we have a product, we use the quotient rule when we have a quotient. We will always use the chain rule. (Actually we've been using it all along and didn't know it!)

The chain rule

In order to prove the chain rule, we will use a different form for the definition of the derivative than we have previously used in proofs. We will use $f'(x) = \lim_{x_1 \to x} \dfrac{f(x_1) - f(x)}{x_1 - x}$.

Let $h(x) = f(g(x))$.

$$\begin{aligned} h'(x) &= \lim_{x_1 \to x} \frac{h(x_1) - h(x)}{x_1 - x} \\ &= \lim_{x_1 \to x} \frac{f(g(x_1)) - f(g(x))}{x_1 - x} \end{aligned}$$

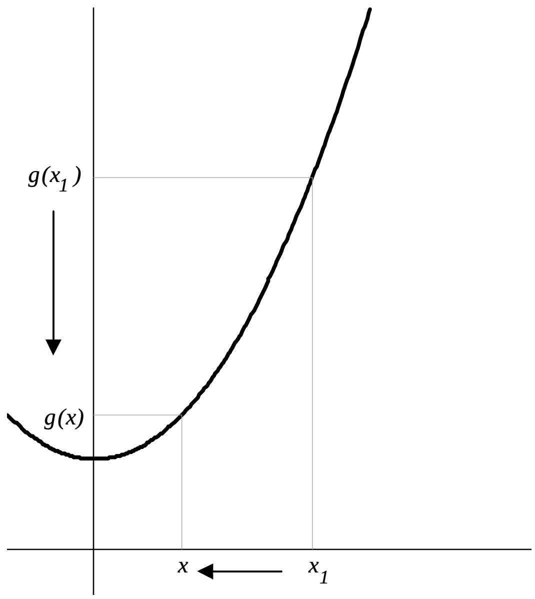
In proving the product and quotient rule we used a technique in which we added zero to change the way an expression looked…and were thereby able to move on with the proof. Here we use another common technique to change the appearance of an expression, we multiplying by one.

$$h'(x) = \lim_{x_1 \to x} \frac{f(g(x_1)) - f(g(x))}{x_1 - x} \bullet \frac{g(x_1) - g(x)}{g(x_1) - g(x)}$$

The commutative property of multiplication allows us to write,

$$h'(x) = \lim_{x_1 \to x} \frac{f(g(x_1)) - f(g(x))}{g(x_1) - g(x)} \bullet \frac{g(x_1) - g(x)}{x_1 - x}$$

Let's consider the first of these two limits. As long as *g* is a well-behaved function, as x_1 approaches x, $g(x_1)$ will approach $g(x)$. The sketch below illustrates this point.



This fact allows us to rewrite the limit as

$$\lim_{g(x_1)\to g(x)} \frac{f(g(x_1))-f(g(x))}{g(x_1)-g(x)}$$ which , if you look carefully, is the definition of $f'(g(x))$!

The second limit, $\lim_{x_1\to x} \frac{g(x_1)-g(x)}{x_1-x}$, is the definition of $g'(x)$.

Putting it all together we obtain

$$h'(x) = f'(g(x)) \bullet g'(x)$$

The Chain Rule

If $h(x) = f(g(x)) \to h'(x) = f'(g(x)) \bullet g'(x)$

or

$D_x[f(g(x))] = f'(g(x)) \bullet g'(x)$

The best way to learn how the chain rule works is to see actual problems.

Consider $h(x) = (7x-2)^{45}$. Now we could, theoretically, expand this binomial and then simply use our power rule (see you in about a year!) Let's instead consider h as a composition in which $f(x) = x^{45}$ and $g(x) = 7x-2$ and $h(x) = f(g(x))$. Since $f(x) = x^{45}$, $f'(x) = 45x^{44}$. Since $g(x) = 7x-2$, $g'(x) = 7$. Because $f'(x) = 45x^{44}$, $f'(g(x)) = 45(7x-2)^{44}$. Thus, $[f(g(x))]' = f'(g(x)) \bullet g'(x) = 45(7x-2)^{44} \bullet 7$

Another way to look at a function like $h(x)=(7x-2)^{45}$ is to "see" an outside and an inside function. In this case the outside function is *somthing*^{45} and the inside function is $7x-2$. To find the derivative, think, "the derivative of *something* to the 45th is 45 times *something* to the 44th times the derivative of the *something*."

When you look at a problem, you should try to "see" the outside function. The following table may help. Don't worry, the more problems you do, the better you'll get!

The problem	**What you should see**	**What you say to yourself**	**The derivative**
$(x^2-6x)^{20}$	*something* to the 20th	20 times *something* to the 19th times the derivative of the *something*	$20(x^2-6x)^{19}(2x-6)$
$\sqrt{\sin x}$	*something* to the 1/2	1/2 times *something* to the minus 1/2 times the derivative of the *something*	$\frac{1}{2}(\sin x)^{-1/2}\cos x$
$\cos^5 x$	*something* to the 5th	5 times *something* to the fourth times the derivative of the *something*	$5\cos^4 x(-\sin x)$
$\sin(5x+4)$	sine of *something*	cosine of *something* times the derivative of the *something*	$[\cos(5x+4)](5)$

We will often have to apply the chain rule to compositions of three or more functions. In this case, work from the outside function to the inside function. We will look at some examples later on.

Changes in our differentiation theorems

Now that we have our chain rule, many of our differentiation theorems will change to reflect the fact that they work for compositions. We will not change the way the product rule or quotient rule look—you just have to remember to chain when necessary.

The Power Rule

$$D_x[f(x)^n]=n\,f(x)^{n-1}f'(x)$$

Derivatives of the trigonometric functions

$$D_x[\sin f(x)]=f'(x)\cos f(x)$$
$$D_x[\cos f(x)]=-f'(x)\sin f(x)$$
$$D_x[\tan f(x)]=f'(x)\sec^2 f(x)$$
$$D_x[\cot f(x)]=-f'(x)\csc^2 f(x)$$
$$D_x[\sec f(x)]=f'(x)\sec f(x)\tan f(x)$$
$$D_x[\csc f(x)]=-f'(x)\csc f(x)\cot f(x)$$

Another form of the chain rule

The chain rule can also be expressed in the following form:

The Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du} \bullet \frac{du}{dx}$$

This form is used to differentiate functions when they are given in terms of " $y =$ " instead of function notation. It will also be of use when we learn how to differentiate implicitly.

Consider $y = \sqrt{x^3 + 5x}$. For this equation, $y = \sqrt{u}$ and $u = x^3 + 5x$. Since $y = \sqrt{u}$, $\frac{dy}{du} = \frac{1}{2}u^{-1/2}$ and since $u = x^3 + 5x$, $\frac{du}{dx} = 3x^2 + 5$. Using $\frac{dy}{dx} = \frac{dy}{du} \bullet \frac{du}{dx}$ yields

$$\frac{dy}{dx} = \left[\tfrac{1}{2}u^{-1/2}\right]\left[3x^2 + 5\right].$$

Substituting $u = x^3 + 5x$ results in

$$\begin{aligned}\frac{dy}{dx} &= \left[\tfrac{1}{2}(x^3 + 5x)^{-1/2}\right]\left[3x^2 + 5\right] \\ &= \frac{3x^2 + 5}{2\sqrt{x^3 + 5x}}\end{aligned}$$

Chain rule examples

Example 1

Given $f(x) = \sqrt{x^2 + 1}$, find $f'(x)$.

$$\begin{aligned}f'(x) &= \tfrac{1}{2}(x^2 + 1)^{-1/2}(2x) \\ &= \frac{2x}{2\sqrt{x^2 + 1}} \\ &= \frac{x}{\sqrt{x^2 + 1}}\end{aligned}$$

Example 2

Given $f(x) = \sin 7x$, find $f'(x)$.

$$\begin{aligned}f'(x) &= (\cos 7x)(7) \\ &= 7\cos 7x\end{aligned}$$

Example 3

Given $g(x) = \sin^2 x$, find $g'(x)$.

$$g'(x) = 2\sin x\cos x$$

(In this problem you should "see" *something* squared.)

Example 4

Given $h(x) = \sin x^2$, find $h'(x)$.

$$\begin{aligned} h'(x) &= (\cos x^2)(2x) \\ &= 2x\cos x^2 \end{aligned}$$

(In this problem you should see the sine of *something*.)

Example 5

Given $f(x) = \tan x^2$, find $f'(x)$.

$$\begin{aligned} f'(x) &= (\sec^2 x^2)(2x) \\ &= 2x\sec^2 x^2 \end{aligned}$$

Example 6

Given $f(x) = \cos^5(3x+5)$, find $f'(x)$.

In this problem we have 3 functions being composed. Think of it like this:

…the derivative of *something* to the 5th which will become 5 times *something* to the 4th
…the derivative of cosine of *something* which will become minus sine of *something*
…the derivative of $3x+5$ which will become 3

$$\begin{aligned} f'(x) &= \left[5\cos^4(3x+5)\right][-\sin(3x+5)][3] \\ &= -15\cos^4(3x+5)\sin(3x+5) \end{aligned}$$

Example 7

Given $f(x) = \dfrac{1}{(x^3-8x^2+3)^6}$, find $f'(x)$.

$$\begin{aligned} f(x) &= (x^3-8x^2+3)^{-6} \\ f'(x) &= -6(x^3-8x^2+3)^{-7}(3x^2-16x) \\ &= \frac{96x-18x^2}{(x^3-8x^2+3)^7} \end{aligned}$$

Example 8

Given $h(x)=\cos^2(\cos x)$, find $h'(x)$.

$$h'(x)=\left[2\cos(\cos x)\right]\left[-\sin(\cos x)\right]\left[-\sin x\right]$$

Example 9

If $h(x)=f(g(x))$ and $f'(7)=4$, $g'(2)=5$ and $g(2)=7$, find $h'(2)$.

$$\begin{aligned} h'(x) &= f'(g(x))g'(x) \\ h'(2) &= f'(g(2))g'(2) \\ &= f'(7)\bullet 5 \\ &= 4\bullet 5 \\ &= 20 \end{aligned}$$

Example 10

Given $p(x)=f(g(h(x)))$, find $p'(x)$.

$$p'(x)=f'(g(h(x)))\bullet g'(h(x))\bullet h'(x)$$

The three types of tangent to a curve problems

Throughout this course we will be writing equations of tangents and normals to curves. A normal is a line perpendicular to the tangent at a given. These problems fall into three basic categories which we will refer to as Category I, II and III:

- Category I: given a curve and a point (or at least an *x*-coordinate)
- Category II: given a curve and a line to which the tangent must be parallel or perpendicular
- Category III: given a curve and a point not on the curve through which the tangent must pass

Category I strategy

To find the point: If you are only given an *x*-coordinate, $x=a$, determine the *y*-coordinate by finding $f(a)$.

To find the slope: Differentiate the function and then calculate $f'(a)$.

Use the point-slope form for a line to write the equation of the tangent.

Example of a Category I tangent problem

Write an equation of a tangent to $y = x^3 + 1$ at $x = 3$.

<u>Point</u>	<u>Slope</u>	<u>Equation of tangent</u>
When $x = 3$, $y = 28$ $\therefore (3, 28)$	$\frac{dy}{dx} = 3x^2$ $\left.\frac{dy}{dx}\right\|_{x=3} = 27$ $\therefore m_T = 27$	$y - 28 = 27(x - 3)$

<u>Category II strategy</u>

To find the point: Differentiate the function and set the derivative equal to the slope of the given line. (If you are writing an equation of a normal, set the derivative equal to the opposite inverse of the slope of the given line.) This will yield an x-value. Now, to get the *y*-coordinate, find the value of the function at this *x*-value.

To find the slope: It is given. If the line given is parallel to the tangent, use the slope of the given line. If the line given is perpendicular to the tangent line, use the opposite inverse of the slope of the given line.

Use the point-slope form for a line to write the equation of the tangent (or normal).

Example of a Category II tangent problem

Write an equation of a tangent to $y = x^3 + 1$ that is parallel to $3x - 4y = 7$.

<u>Slope</u>

The slope of the given line is $\frac{3}{4}$,

thus the slope of our tangent is $\frac{3}{4}$.

<u>Point</u>

$$\frac{dy}{dx} = 3x^2$$

$$\therefore 3x^2 = \frac{3}{4}$$

$$x = \frac{1}{2} \text{ or } x = -\frac{1}{2}$$

This means there are two lines tangent to the curve that are parallel to the given line.

When

$x = \frac{1}{2} \rightarrow y = \frac{9}{8} \therefore \left(\frac{1}{2}, \frac{9}{8}\right)$ is one point.

$x = -\frac{1}{2} \rightarrow y = \frac{7}{8} \therefore \left(-\frac{1}{2}, \frac{7}{8}\right)$ is another point.

Equations of tangents

$$y - \tfrac{9}{8} = \tfrac{3}{4}\left(x - \tfrac{1}{2}\right)$$

and

$$y - \tfrac{7}{8} = \tfrac{3}{4}\left(x + \tfrac{1}{2}\right)$$

Category III strategy

To find the point: Name an arbitrary point on the curve. If the curve is $y = x^3 + 1$, the point would be $(x, x^3 + 1)$. Find the slope of the line through the given point and this arbitrary point. Set the derivative equal to this slope and solve for *x*. Now, to get the *y*-coordinate, find the value of the function at this *x*-value. You will often find more than one point.

To find the slope: Find the value of the derivative at the point(s) you just found.

Use the point-slope form for a line to write the equation of the tangent.

Example of a Category III tangent problem

Write an equation(s) of the tangent(s) to $y = x^3 + 1$ passing through the point $(3,1)$.

Point

Any point on the curve can be described by $(x, x^3 + 1)$. The slope of the line through $(x, x^3 + 1)$ and $(3,1)$ is given by

$$m = \frac{(x^3 + 1) - 1}{x - 3} = \frac{x^3}{x - 3}$$

The derivative is $\dfrac{dy}{dx} = 3x^2$. Setting $\dfrac{dy}{dx} = m$ yields

$$\frac{x^3}{x - 3} = 3x^2$$

$$x = \tfrac{9}{2} \text{ or } x = 0$$

When $x = \tfrac{9}{2} \rightarrow y = \tfrac{737}{8} \therefore$ one point is $\left(\tfrac{9}{2}, \tfrac{737}{8}\right)$

$x = 0 \rightarrow y = 1 \therefore$ another point is $(0,1)$.

Slope of tangent

$$\left.\frac{dy}{dx}\right|_{x=\frac{9}{2}} = \frac{243}{4} \text{ and } \left.\frac{dy}{dx}\right|_{x=0} = 0$$

Equations of tangents

$$y - \frac{737}{8} = \frac{243}{4}\left(x - \frac{9}{2}\right)$$

and

$$y - 1 = 0(x - 0)$$

Differentiability and Continuity

Introduction

Like continuity, differentiability can be seen as a measure of a curve's smoothness. As you will soon see, a curve that is differentiable is even more smooth, more well-behaved than a function that is continuous but not differentiable. When the question, "Is *f* differentiable at $x = a$?" is asked, all we want to know is whether or not the derivative exists at $x = a$. Take, for example, $f(x) = \sqrt[5]{x}$. Its derivative, $f'(x) = \dfrac{1}{5\sqrt[5]{x^4}}$, clearly does not exist at $x = 0$. We say that *f* is not differentiable at $x = 0$.
Many times questions about differentiability are this simple…but not always. There are three situations in which a function can fail to be differentiable at $x = a$:

I. the function may be discontinuous at $x = a$
II. it may have a vertical tangent at $x = a$
III. it may have a cusp at $x = a$.

The continuity problem

Consider the following function.

$$f(x) = \begin{cases} x^2 + 3 & \text{if } x \le 2 \\ x^2 - 4 & \text{if } x > 2 \end{cases}.$$

The derivative of *f* is

$$f'(x) = \begin{cases} 2x & \text{if } x < 2 \\ 2x & \text{if } x > 2 \end{cases}$$

Note that the "less than or equal to" symbol has been replaced with the strict inequality $x < 2$. This is because we don't know if the derivative exists at $x = 2$. If we leave the "less than or equal to" symbol in place, we are already saying the derivative exists at $x = 2$…the very thing we're trying to determine. Now, to find the value of the derivative at $x = 2$, we will have to find a left-hand and right-hand derivative. This should not be surprising since a derivative is a limit—and finding the limit of a piecewise function requires both left- and right-hand limits.
Now, $f_+'(2) = 4$ and $f_-'(2) = 4$ so it would appear that *f* is differentiable at $x = 2$… $f'(2)\ \exists$. Before we make any conclusion however, we must run a continuity test at $x = 2$.

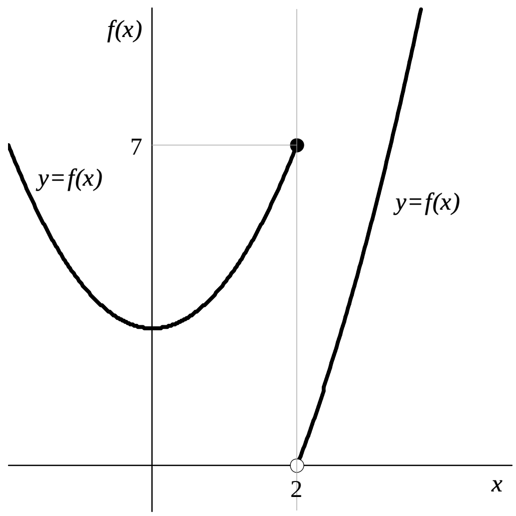
<u>Continuity test at $x = 2$</u>

$$f(2) = 7$$

$$\left.\begin{aligned} \lim_{x\to 2^+} f(x) = 7 \\ \lim_{x\to 2^-} f(x) = 0 \end{aligned}\right\} \lim_{x\to 2} f(x)\ \nexists$$

Since $\lim_{x\to 2} f(x)\ \nexists$, *f* is discontinuous at $x = 2$.

If we look at the graph of *f* below we'll see what's happening.

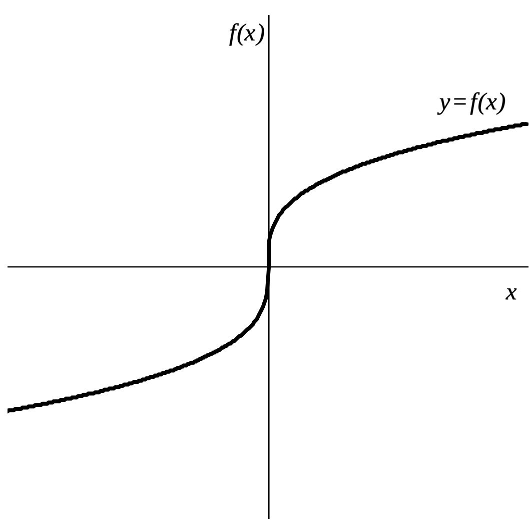


Although both the left- and right-hand derivatives are equal, they occur at different points on the curve! Since *f* is discontinuous at $x = 2$, *f* is not differentiable at $x = 2$.

Consider $f(x) = \frac{1}{x}$. Clearly *f* does not exist at $x = 0$ and thus is not continuous at $x = 0$. The derivative of *f* is $f'(x) = -\frac{1}{x^2}$ which also fails to exist at $x = 0$.
In general, we can say if a function is discontinuous at $x = a$, the function is not differentiable at $x = a$.

The vertical tangent

Consider the function $f(x) = \sqrt[3]{x}$ and its derivative $f'(x) = \frac{1}{3\sqrt[3]{x^2}}$. The function *f* is continuous at $x = 0$, but the derivative does not exist at $x = 0$. Since the derivative does not exist at $x = 0$, *f* is not differentiable at $x = 0$…even though it is continuous at $x = 0$. So what's happening at $x = 0$? Take a look at the graph of *f*.

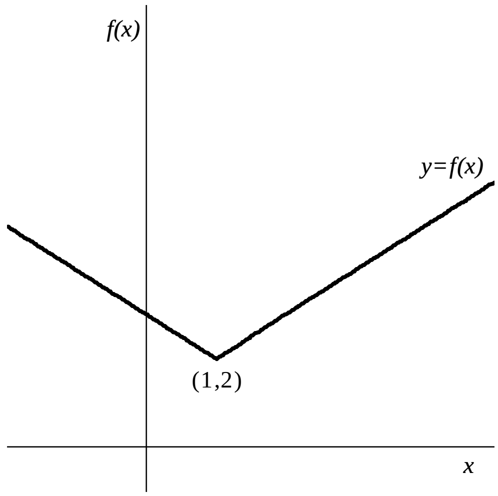


This curve has a vertical tangent at $x = 0$!

We know that if a function is NOT continuous, it is NOT differentiable. Here we have a case in which a function is continuous but not differentiable…so continuity does not imply differentiability. Another way to say this is that ***continuity is a necessary but insufficient condition for differentiability***.

The cusp

The third situation in which a function may fail to be differentiable involves something called a cusp. Consider $f(x)=|x-1|+2$.



This curve has a cusp at the point (1,2).

Certainly, *f* is continuous at $x=1$, but is it differentiable? Let's take a look at its derivative. (Remember, functions involving absolute value are piecewise functions and must be converted to piecewise before taking a derivative!)

$$f(x)=|x-1|+2=\begin{cases} x-1+2 & \text{if } x\geq 1 \\ 1-x+2 & \text{if } x<1 \end{cases}=\begin{cases} x+1 & \text{if } x\geq 1 \\ -x+3 & \text{if } x<1 \end{cases}$$

Thus,

$$f'(x)=\begin{cases} 1 & \text{if } x>1 \\ -1 & \text{if } x<1 \end{cases}$$

Now, $f_+'(1)=1$ but $f_-'(1)=-1$, therefore $f'(1)$ does not exist and *f* is not differentiable at $x=1$. Again, we have a function which is continuous at $x=a$ but is not differentiable at $x=a$.

Any function that has a cusp at $x=a$ will not be differentiable at $x=a$.

Relationship between continuity and differentiability

If a function is differentiable at $x=a$, it is continuous at $x=a$. *Differentiability implies continuity.*
If a function is continuous at $x=a$, it may or may not be differentiable at $x=a$.
If a function is not differentiable at $x=a$, it may or may not be continuous at $x=a$.
If a function is not continuous at $x=a$, it is not differentiable at $x=a$. *Not continuous implies not differentiable.*

The fact that differentiability implies continuity gives us another test for continuity at a number for non-piecewise functions. Instead of running our usual continuity test at $x = a$, we can just take the derivative and determine if the derivative exists at $x = a$. If it does, then the function is continuous at $x = a$. Be careful though…if the derivative does <u>not</u> exist at $x = a$, it does <u>not necessarily mean</u> the function is discontinuous at $x = a$.

Example 1

Given $f(x) = \begin{cases} x^2 + 1 & \text{if } x \le 1 \\ 2x & \text{if } x > 1 \end{cases}$, determine if *f* is differentiable at $x = 1$.

Because *f* is a piecewise function, we will first perform a continuity test at $x = 1$.

<u>Continuity test at *x* = 1</u>

$$f(1) = 2$$

$$\left.\begin{matrix} \lim_{x \to 1^+} f(x) = 2 \\ \lim_{x \to 1^-} f(x) = 2 \end{matrix}\right\} \therefore \lim_{x \to 1} f(x) = 2$$

Since $\lim_{x \to 1} f(x) = f(1)$, f is continuous at $x = 1$

We will now check for differentiability.

$$f'(x) = \begin{cases} 2x & \text{if } x > 1 \\ 2 & \text{if } x < 1 \end{cases}$$

$$f_+'(1) = 2 \text{ and } f_-'(1) = 2$$

Since $f_+'(1) = f_-'(1)$, f is differentiable at $x = 1$.

Example 2

Given $f(x) = \begin{cases} x^2 + 2 & \text{if } x \le 1 \\ x + 2 & \text{if } x > 1 \end{cases}$, determine if *f* is differentiable at $x = 1$.

Because *f* is a piecewise function, we will first perform a continuity test at $x = 1$.

<u>Continuity test at *x* = 1</u>

$$f(1) = 3$$

$$\left.\begin{matrix} \lim_{x \to 1^+} f(x) = 3 \\ \lim_{x \to 1^-} f(x) = 3 \end{matrix}\right\} \therefore \lim_{x \to 1} f(x) = 3$$

Since $\lim_{x \to 1} f(x) = f(1)$, f is continuous at $x = 1$.

We will now check for differentiability.

$$f'(x) = \begin{cases} 2x & \text{if } x > 1 \\ 1 & \text{if } x < 1 \end{cases}$$

$$f_+'(1) = 2 \text{ and } f_-'(1) = 1$$

Since $f_+'(1) \ne f_-'(1)$, f is not differentiable at $x = 1$.

Example 3

Given $f(x)=\begin{cases} x^2 & \text{if } x<1 \\ ax+b & \text{if } x\ge 1 \end{cases}$, find the values of a and b so that f is differentiable at $x=1$.

We are being asked to find the value of two constants, so we need to find two equations in two variables and solve the system. Where will we get the two equations? One from a continuity test and one from the derivative.

Continuity test at $x = 1$

$$f(1)=a+b$$

$$\lim_{x\to 1^+} f(x)=a+b$$

$$\lim_{x\to 1^-} f(x)=1$$

For f to be continuous at $x=1$,

$$a+b=1$$

The derivative at x = 1

$$f'(x)=\begin{cases} 2x & \text{if } x<1 \\ a & \text{if } x>1 \end{cases}$$

Now, $f_+'(1)=a$ and $f_-'(1)=2$.

For f to be differentiable at $x=1$,

$$a=2$$

Substituting $a=2$ into $a+b=1$ yields $b=-1$.

Therefore, for f to be differentiable at $x=1$, $a=2$ and $b=-1$.

Example 4

Determine if $f(x)=\dfrac{x}{x-5}$ is differentiable at $x=5$.

Since $f(5)$ does not exist, f is not continuous at $x=5$, thus f is not differentiable at $x=5$.

Example 5

Determine if $g(x)=\sqrt{x+6}$ is differentiable at $x=10$.

Since $g'(x)=\dfrac{1}{2\sqrt{x+6}}$, $g'(10)=\frac{1}{8}$. Since $g'(10)$ exists, g is differentiable at $x=10$.

Higher Order Derivatives

Introduction

This particular topic actually contains very little in terms of new concepts although the algebra at times may seem a bit tedious. A "higher order derivative" is simply the derivative of a derivative…or the derivative of the derivative of a derivative…and so on. If you have a function *f* and find its derivative, you have found f', the first derivative. If you take the derivative of f', you would have f'', the second derivative. For many functions this process can go on forever—for other functions the process of taking higher and higher order derivatives eventually reaches a point where the derivative is zero or cycles back to the original function. We'll see examples of each.

In the course of most of our problems, the second derivative is as far as we will need to go.

Notation

The table below shows the most commonly used notations for higher order derivatives.

	First derivative	Second derivative	Third derivative	*n*th derivative
$g(x)$	$g'(x)$	$g''(x)$	$g'''(x)$	$g^n(x)$
y	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$	$\frac{d^3y}{dx^3}$	$\frac{d^ny}{dx^n}$
	$D_x[f(x)]$	$D_x^2[f(x)]$	$D_x^3[f(x)]$	$D_x^n[f(x)]$

Some interesting situations

Consider $f(x) = x^3 + 3x^2 - 2x + 6$. Let's repeatedly differentiate *f*.

$$f'(x) = 3x^2 + 6x - 2$$
$$f''(x) = 6x + 6$$
$$f'''(x) = 6$$
$$f^4(x) = 0$$

Now let's try it again with $g(w) = 2w^4 + w^3 - 5w^2$.

$$g'(w) = 8w^3 + 3w^2 - 10w$$
$$g''(w) = 24w^2 + 6w - 10$$
$$g'''(w) = 48w + 6$$
$$g^4(w) = 48$$
$$g^5(w) = 0$$

Repeatedly differentiating a polynomial will eventually lead to a derivative that is zero. In the first example above we began with a third order polynomial and the fourth derivative was zero. In the second example, we began with a fourth order polynomial and the fifth derivative was zero. As you've probably guessed already, if we begin with an nth order polynomial, the $(n+1)$th derivative will always be zero.

Let's turn our attention to another interesting scenario. Consider $f(x)=\sin x$. Repeatedly differentiating results in the following:

$$f'(x)=\cos x$$
$$f''(x)=-\sin x$$
$$f'''(x)=-\cos x$$
$$f^4(x)=\sin x$$

Clearly, the derivatives repeat in a cycle of four derivatives. This means it is very easy to find a very high order derivative of $\sin x$. Suppose we wanted $D_x^{26}[\sin x]$. Since we know every derivative that is a multiple of four is $\sin x$, we know $D_x^{24}[\sin x]=\sin x$. All we need to do now is differentiate two more times which yields $D_x^{26}[\sin x]=-\sin x$. Another way to look at the same problem is to ask, "What is the remainder when 26 is divided by 4?" The remainder is two, so the twenty-sixth derivative is the same as the second derivative!

The same cyclic behavior occurs with $\cos x$. If $f(x)=\cos x$,

$$f'(x)=-\sin x$$
$$f''(x)=-\cos x$$
$$f'''(x)=\sin x$$
$$f^4(x)=\cos x$$

Do all six trigonometric functions exhibit this same cyclic behavior? We leave it to you, the student, to experiment on your own.

Example 1

Given $y=\frac{1}{x}$, determine $\frac{d^3y}{dx^3}$.

$$y=x^{-1}$$
$$\frac{dy}{dx}=-x^{-2}$$
$$\frac{d^2y}{dx^2}=2x^{-3}$$
$$\frac{d^3y}{dx^3}=-6x^{-4}=-\frac{6}{x^4}$$

Example 2

Given $g(x) = (3x+5)^5$, determine $g''(x)$.

$$g(x) = (3x+5)^5$$
$$g'(x) = 5(3x+5)^4(3)$$
$$= 15(3x+5)^4$$
$$g''(x) = 60(3x+5)^3(3)$$
$$= 180(3x+5)^3$$

Example 3

Given $f(x) = \sin x$, determine $f^{217}(x)$.

217 divided by 4 leaves a remainder of 1 so the 217th derivative is the same as the 1st.

$$f(x) = \sin x$$
$$f'(x) = \cos x$$
$$\therefore f^{217}(x) = \cos x$$

Example 4

Given $h(x) = \dfrac{x-2}{3x+4}$, determine $h''(x)$.

$$h(x) = \frac{x-2}{3x+4}$$
$$h'(x) = \frac{(3x+4)(1)-(x-2)(3)}{(3x+4)^2}$$
$$= \frac{10}{(3x+4)^2}$$

Now, at this point we can use the quotient rule again or rewrite $h'(x)$ and use the power rule. Here we choose to rewrite $h'(x)$ and use the power rule.

$$h'(x) = 10(3x+4)^{-2}$$
$$h''(x) = -20(3x+4)^{-3}(3)$$
$$= -\frac{60}{(3x+4)^3}$$