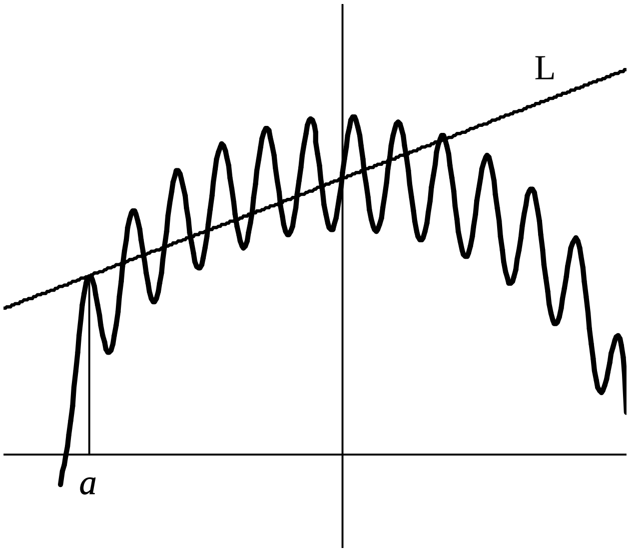
Tangents and the Velocity Problem

Introduction

This section is actually a preview of what's to come…and a history lesson. The final development of Calculus in the 17th century was centered around several problems from both the applied and pure branches of mathematics. One problem involved finding a way to determine the instantaneous velocity of an object. Mathematicians realized that this was closely related to the problem of finding the slope of a tangent to a curve. The other problem was actually an ancient one…finding the exact volume of an arbitrary solid. Two thousand years ago, Archimedes worked on, and made significant progress in determining volumes of solids. We will actually use his method later on in the course. In this section we will discuss the velocity/tangent problem.

The techniques we will use in this section will not give us exact answers. We will be using fairly "ugly" methods to estimate the slope of a tangent to a curve. Later on, after we know how to work with "limits" we will be able to find the exact slope of a tangent to a curve at a given point.

We've used the word "tangent" several times now. In geometry, you studied tangents—primarily tangents to circles. In this course, when we speak of a line being tangent to a curve we mean tangent at a point—what the tangent line does before or after does not matter to us. In geometry you learned that a tangent touches a circle at exactly one point. In calculus, you will see many tangents that intersect curves…sometimes many times. Consider the diagram below.



The line L is tangent to the curve at $x = a$. Notice that L intersects the curve at several other points.

Approximating a tangent when the function is given in algebraic form

Let's try to write an equation for a line that is tangent to the graph of $f(x)=x^2$ at $x=1$. A tangent line is, of course, just a line. All we ever need to write the equation of a line is a point and the slope. The point is easy. Since we are given $x=1$, we find $f(1)=1$ and so the point is $(1,1)$.

We will not be able to find the exact slope of this tangent. Not yet. Not until we know about limits and derivatives. What we will do in this and the next section is find ways to <u>approximate</u> a tangent line.

Before we work on finding the slope of a tangent line, we need to know what a secant line is. A secant line is a line that intersects, but is not tangent to, a curve. To begin estimating the slope of the tangent we will find the slope of a secant line that passes through $(1,1)$ and another point Q on the curve close to $(1,1)$. We will continue to find the slope of this secant line as Q gets closer and closer to $(1,1)$. In other words, we will let Q approach $(1,1)$. Now, there are two ways to approach the point $(1,1)$ --from the left and from the right—we'll do both.

Instead of choosing Q to be $(1.5, 2.25)$, calculating a slope, then letting Q be $(1.1, 1.21)$, calculating a slope and so on, let's just let Q be an arbitrary point on $f(x)=x^2$. Q's coordinates are then (x, x^2). Now, we can generate a general expression for the slope of our secant that passes through (x, x^2) and $(1,1)$. The slope will always be $\frac{x^2-1}{x-1}$. The table below shows the resulting secant line slope for several values of x…letting Q approach $(1,1)$ from the right.

x	$m=\frac{x^2-1}{x-1}$
1.5	2.5
1.1	2.1
1.01	2.01
1.001	2.001

Notice that as Q gets closer and closer to the point $(1,1)$ from the right, the slope of the secant line gets closer and closer to 2.

The slope of the secant as Q approachs $(1,1)$ from the left is shown below.

x	$m=\frac{x^2-1}{x-1}$
.5	1.5
.9	1.9
.99	1.99
.999	1.999

As Q gets closer and closer to the point $(1,1)$ from the left, the slope of the secant line again gets closer and closer to 2.

Since we have approached $(1,1)$ from both the left and right, we can say that as Q gets closer to the point $(1,1)$, the slope of the secant line approaches 2.
In essence what we are saying is that as Q gets closer to $(1,1)$, the tangent at $(1,1)$ and the secant line through $(1,1)$ get closer to becoming the same line. Since the slope of this line is 2 (or so we estimate), the equation of the tangent becomes $y-1=2(x-1)$.

Approximating a tangent given a table of function values

Many times we will not be given a function in algebraic form. Instead we will be given a selection of function values in table form. Consider the following function.

x	0.0	0.1	0.2	0.3	0.4	0.5
$f(x)$	3.86	3.71	3.40	3.02	2.35	1.46

Again, we want to write an equation of a tangent—this time at the point $(0.2, 3.40)$. We cannot use the same procedure we used in the previous problem because we do not have the actual function in algebraic form—so we cannot write an expression for the slope of a secant.

We have our point, what we need is the slope—or at least an approximation of the slope. In this situation, we have three choices.

Choice 1: We can find the slope of the secant line that passes through the point $(0.2, 3.40)$ and the point just to the left of it, (0.1, 3.71). Since the point (0.1, 3.71) is close to the point where we want our tangent, we can use the slope of the secant as an approximation for the slope of the tangent. The slope of this secant line is -3.1. Thus we can say that the slope of the tangent at $(0.2, 3.40)$ is approximately -3.1. Our tangent line is then $y-3.40=-3.1(x-0.2)$.

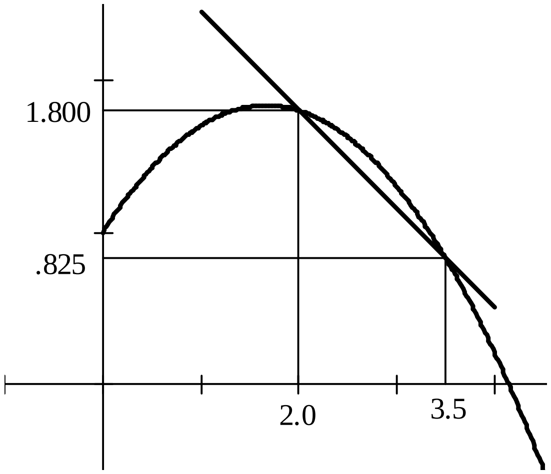
Choice 2: We can find the slope of the secant line that passes through the point $(0.2, 3.40)$ and the point just to the right of it, (0.3, 3.02). Since the point (0.3, 3.02) is close to the point where we want our tangent, we can use the slope of the secant as an approximation for the slope of the tangent. The slope of this secant line is -3.8. Thus we can say that the slope of the tangent at $(0.2, 3.40)$ is approximately -3.8. Our tangent line is then $y-3.40=-3.8(x-0.2)$.

Choice 3: We can use what is called the "symmetric difference". In this case, we use the slope of the secant through the points on either side of $(0.2, 3.40)$ as an approximation for the slope of our tangent line. The slope of this secant, passing through (0.1, 3.71) and (0.3, 3.02), is -3.450. Our tangent line becomes $y-3.40=-3.45(x-0.2)$.

Any of these three lines would serve as an approximation of the tangent line at (0.2, 3.40)

Velocity and the secant line/tangent line

Average velocity over an interval of time is given by $\dfrac{\text{change in position}}{\text{change in time}}$. It is a rate of change. If we consider the graph of a position function $s(t)$, the average velocity over an interval is the slope of the secant line passing through the endpoints of the interval. In the diagram below, the graph of a position function $s(t) = -0.3t^2 + t + 1$ is given. The distance units are feet and the time units are seconds. The average velocity on the interval $[2.0, 3.5]$ can be calculated by finding the slope of the secant line through $(2.0, 1.800)$ and $(3.5, 0.825)$



The slope of the secant line is -0.65 and so the average velocity on $[2.0, 3.5]$ is -0.65 feet per second.

What may be more interesting to know is the velocity at $t = 2.0$. The velocity of an object at a specific value of t is called the <u>instantaneous velocity</u>—instantaneous rates of change will be one of the central concepts of this course.

To estimate the instantaneous velocity of the object at $t = 2.0$, we will consider shorter and shorter time intervals. Since the average velocity is calculated by $\dfrac{\text{change in position}}{\text{change in time}}$ or $\dfrac{s(t_2) - s(t_1)}{t_2 - t_1}$, we can calculate average velocities for our problem by using $v_{avg} = \dfrac{(-0.3t^2 + t + 1) - 1.800}{t - 2.0}$ where the 2.0 is t_1 and the 1.800 is $s(t_1)$. The table below shows average velocities for shorter and shorter time intervals (all starting with $t = 2.0$).

	$\frac{(-0.3t^2+t+1)-1.800}{t-2.0}$
[2.0, 3.5]	–0.650
[2.0, 3.0]	–0.500
[2.0, 2.5]	–0.350
[2.0, 2.3]	–0.290
[2.0, 2.1]	–0.230
[2.0, 2.01]	–0.203
[2.0, 2.001]	–0.200

We can now say that a reasonable estimate of the instantaneous velocity at $t = 2.0$ is -0.200.

We've done quite a bit of estimating in this section. Estimation and approximation are recurring themes in this course. Using the slope of a secant to estimate the slope of a tangent is something we will do quite often. Keep in mind that the slope of a secant is a measure of the average rate of change in a function and the slope of a tangent is a measure of the instantaneous rate of change. We will, of course, move beyond estimations and will be able to determine the exact slope of a tangent to a curve at any point—with relative ease. We will also see that finding instantaneous velocity and finding the slope of a tangent are actually the same problem. In order to go beyond estimating the slope of a tangent, we will need something called the "derivative". The derivative is a limit…so that will be our next step…learning out how to find the limit of a function.

The Limit of a Function

Introduction

We will study two basic types of limits. The first, and the topic of this section, describes the behavior of a function as the independent variable approaches a specific number. By convention, the independent variable we normally use is *x*, so we will refer to these types of limits as "limits as $x \to a$". The notation $x \to a$ is read "*x* approaches *a*". The second type of limit describes the behavior of a function as the independent variable increases or decreases without bound. We will refer to these limits as "limits as $x \to \pm\infty$" or "limits at infinity". In this section we will study limits as $x \to a$. The other category, limits at infinity, will come later.

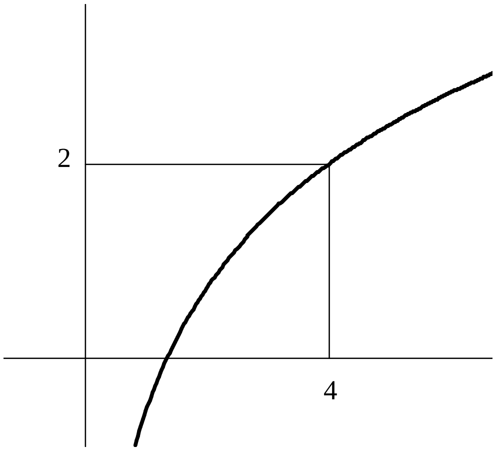
If we want to ask the question, "What happens to the function $f(x)$ as $x \to a$?" we use the following notation: $\lim\limits_{x \to a} f(x)$. When we approach a number, we must approach from both the left and the right. If we are approaching from the right, the limit is called a right-hand limit. If, for instance, we are approaching $x = 5$ from the right, we would find the value of the function for *x* values like 5.200, 5.100, 5.010, 5.001 and so on. A right-hand limit is denoted $\lim\limits_{x \to a^+} f(x)$. Similarly, a left-hand limit is denoted $\lim\limits_{x \to a^-} f(x)$ and we would use values of *x* like 4.800, 4.900, 4.990, and 4.999.

When we attempt to find a limit, we must approach *a* from both directions and the right-hand limit must be equal to the left-hand limit for *the* limit to exist.

$$\boxed{\lim_{x \to a} f(x) = L \text{ if and only if } \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L}$$

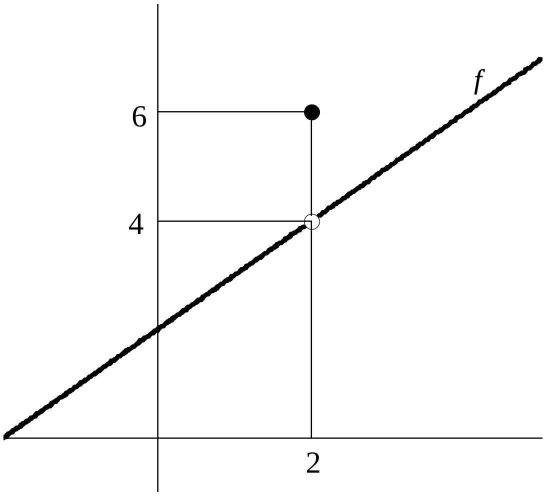
Finding limits from graphs

Consider the following diagram of a function *f*.

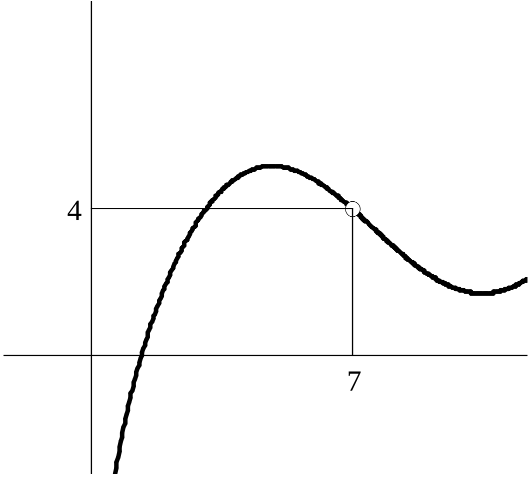


Note that as x gets closer and closer to 4 from the right, the value of the function approaches 2. As x gets closer and closer to 4 from the left, the value of the function also approaches 2. Since $\lim_{x\to 4^+} f(x) = 2$ and $\lim_{x\to 4^-} f(x) = 2$ we can say $\lim_{x\to 4} f(x) = 2$.

You may be asking, "Why not just find $f(4)$? After all $f(4) = 2$." To answer this, let's look at another function.

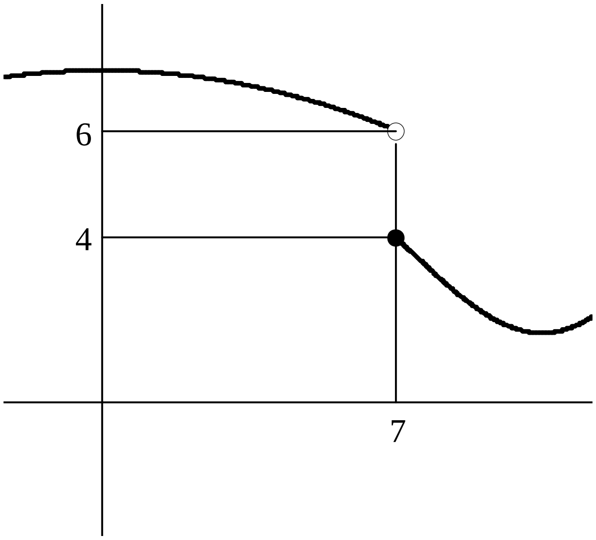


The first thing to note is that $f(2) = 6$. On the other hand $\lim_{x\to 2^+} f(x) = 4$ and $\lim_{x\to 2^-} f(x) = 4$. This means that $\lim_{x\to 2} f(x) = 4$. This is a very important lesson…finding a limit is NOT the same as finding a function value! On occasion, the two will be the same. If they are the same it will mean something important—but that comes later. For now it is vital that you understand that a limit and a function value are not the same thing. As a matter of fact, a function doesn't even have to exist at a particular number to have a limit at that number. Below is the graph of a function *f*.



Notice that $f(7)$ does not exist but $\lim_{x\to 7} f(x) = 4$.

Consider the following graph. This situation illustrates that left- and right-hand limits are not always the same.



For this piecewise function, $\lim_{x\to 7^+} f(x)=4$ but $\lim_{x\to 7^-} f(x)=6$ therefore $\lim_{x\to 7} f(x)$ does not exist.

Finding limits when given an algebraic expression

As you will soon know, there are very simple techniques for determining the limit of a function when the function is given to us algebraically. For now, to find a limit as $x\to a$ we will evaluate the expression at x values that are increasingly close to, but never equal to, a. Remember, we must approach a from both the left and the right!

Consider the following limit: $\lim_{x\to 1}\frac{x-1}{x^2-1}$. Notice first that the expression does not exist at $x=1$. This is of no concern to us because finding a limit at a number has little to do with finding a function value at a number. The table below shows the value of $\frac{x-1}{x^2-1}$ at several values of x approaching $x=1$ from the right and left.

From right	
x	$\frac{x-1}{x^2-1}$
1.500	0.400
1.300	0.453
1.100	0.476
1.010	0.498
1.001	0.500

From left	
x	$\frac{x-1}{x^2-1}$
0.500	0.667
0.800	0.556
0.900	0.526
0.990	0.503
0.999	0.500

From the charts it appears that $\lim_{x\to 1^+}\frac{x-1}{x^2-1}=0.500$ and $\lim_{x\to 1^-}\frac{x-1}{x^2-1}=0.500$ and therefore we will say

$$\lim_{x\to 1}\frac{x-1}{x^2-1}=0.500.$$

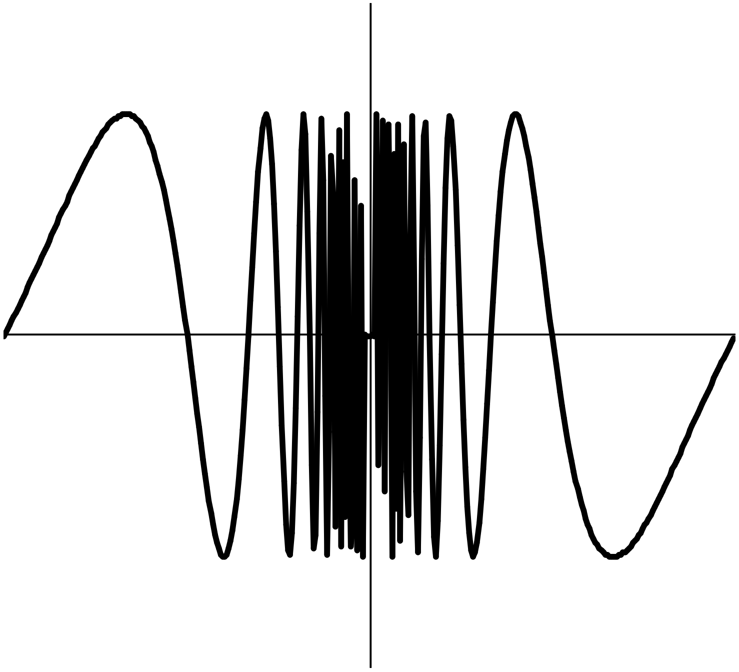
You need to understand that we are actually estimating here. It ***appears*** from the charts that $\lim_{x\to 1}\frac{x-1}{x^2-1}=0.500$ and for now that's the best we can do. Also, we are not saying that at $x=1,\ \frac{x-1}{x^2-1}=.500$. That would not be correct. All we are saying is that as *x* gets closer and closer to 1, the value of the expression gets closer to 0.500.

Estimating a limit using tables has its limitations—it doesn't work nicely for some functions. Let's try to estimate $\lim_{x\to 0}\left[\sin\frac{p}{x}\right]$ using tables of values.

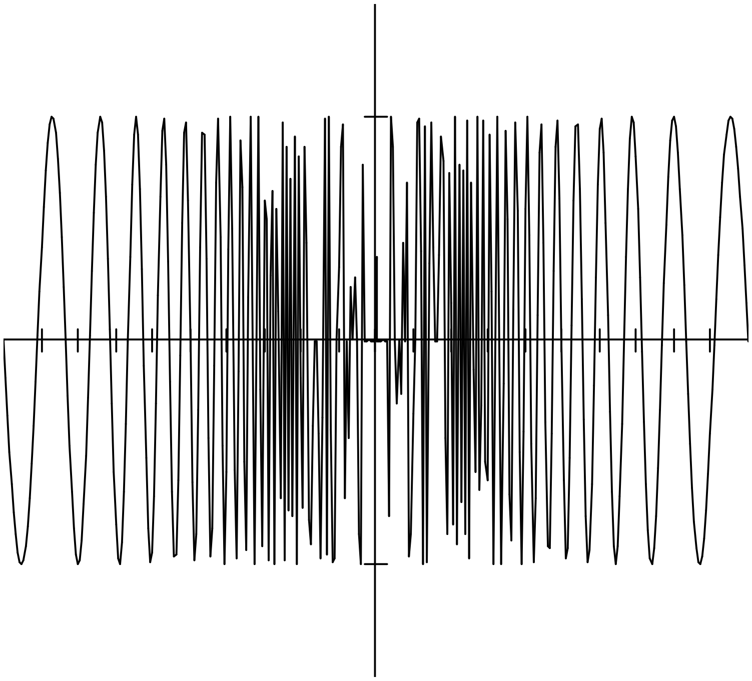
From right	
x	$\sin\frac{p}{x}$
0.500	0.000
0.300	0.866
0.118	1
0.100	0.000
0.010	0.000
0.009	–1
0.001	0.000

From left	
x	$\sin\frac{p}{x}$
–0.500	0.000
–0.300	0.866
–0.118	–1
–0.100	0.000
–0.010	0.000
–0.009	1
–0.001	0.000

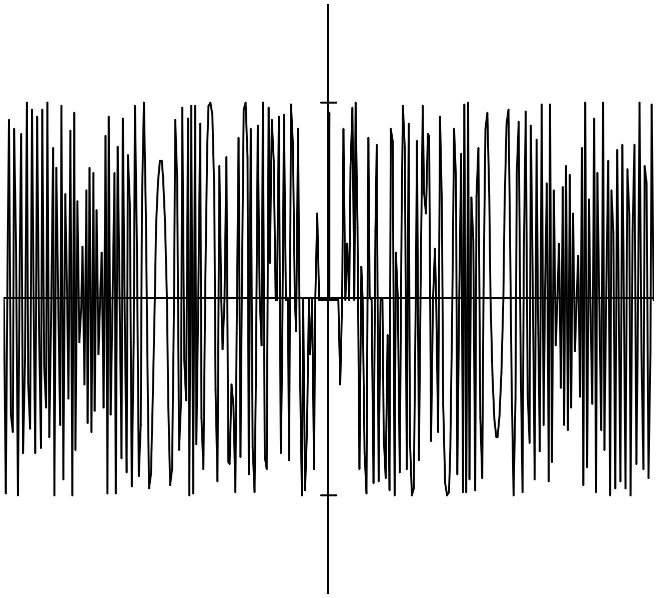
The values of the expression do not appear to nicely approach any particular value and so our charts tell us very little. In fact, $f(x)=\sin\frac{p}{x}$ is a rather fascinating function for which $\lim_{x\to 0}\left(\sin\frac{p}{x}\right)$ does not exist. The graph of *f* on the interval [-1,1] is shown below.



Again, the graph of *f*, but this time on the interval $[-0.100, 0.100]$.



One more time (since it's so crazy!)…this time on [–0.01, 0.01].



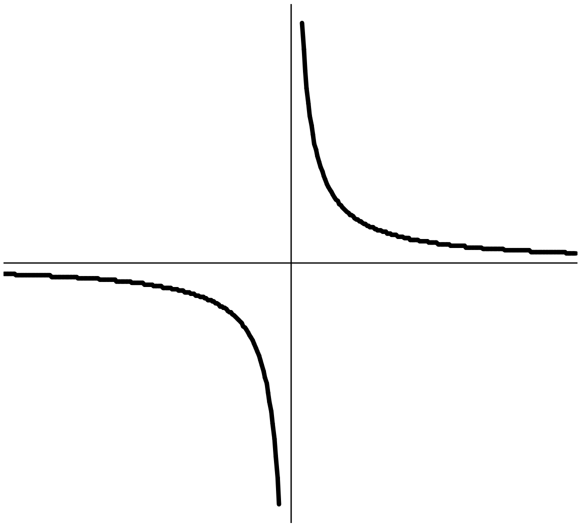
This function oscillates wildly between –1 and 1 and in fact has an infinite number of zeros!

OK…back to work.

We will see a veritable host of piecewise functions this year. When taking the limit of a piecewise function, we need to pay close attention to where the limit is being taken and where each piece is defined. Consider $v(t)=\begin{cases}0 \text{ if } t<0\\1 \text{ if } t\geq 0\end{cases}$. Let's try to find $\lim\limits_{t\to 0}v(t)$. We will need to find both the left- and right-hand limits. When we approach zero from the right, we will use the bottom piece and when approaching from the left we will use the top piece. Now, $\lim\limits_{t\to 0^+}v(t)=1$ but $\lim\limits_{t\to 0^-}v(t)=0$ so $\lim\limits_{x\to 0}v(t)$ does not exist. Notice though, if we were asked to find $\lim\limits_{x\to 7}v(t)$ we would use the bottom piece when approaching from either direction and $\lim\limits_{t\to 7}v(t)=1$.

Non-existent limits

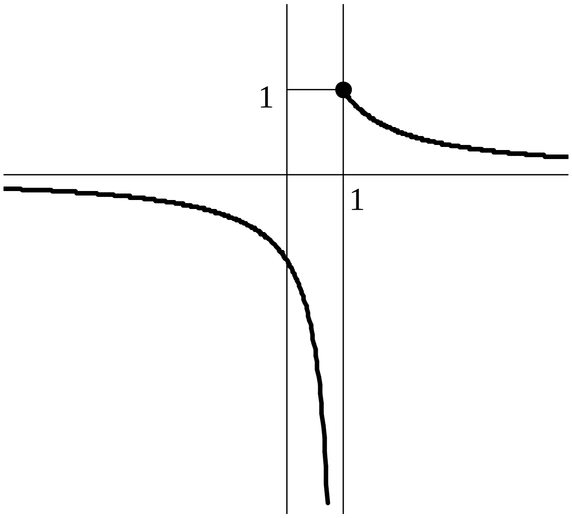
We've already seen several cases where a limit can fail to exist. In most of the previous examples, when a limit failed to exist, it did so because the left- and right-hand limits were not equal. This is not the only situation in which a limit can fail to exist. Let's take a look at $f(x)=\dfrac{1}{x}$ and the limit of *f* as $x\to 0$. The graph of *f* is shown below.



As $x\to 0^+$, the function values increase without bound. We say the function "blows up". Since the function takes on larger and larger function values, it literally has no bound—no limit. We denote this by $\lim\limits_{x\to 0^+}f(x)=\infty$. Stating $\lim\limits_{x\to 0^+}f(x)=\infty$ is the same as saying the limit does not exist. By stating "$=\infty$" we are simply giving the reader additional information about how the limit fails to exist. If we approach zero from the left we obtain $\lim\limits_{x\to 0^-}f(x)=-\infty$. This means that as we approach zero from the left, the function values decrease without bound—the function "blows down". Again, the limit fails to exist.

The limit $\lim\limits_{x\to 0}f(x)$ then fails to exist not because the left- and right-hand limits were not equal but because the individual limits from each side failed to exist. Let's alter the problem just a little and consider one last example.

Consider the function $f(x)=\begin{cases}\dfrac{1}{x} & \text{if } x\geq 1\\ \dfrac{1}{x-1} & \text{if } x<1\end{cases}$ and the limit $\lim\limits_{x\to 1} f(x)$. Its graph is show below.



Now, $\lim\limits_{x\to 1^+} f(x)=1$ and $\lim\limits_{x\to 1^-} f(x)=-\infty$. This means that $\lim\limits_{x\to 1} f(x)$ does not exist. Again, it fails to exist not because the left- and right-hand limits are not equal, but because the left-handed limit itself does not exist.

There are then two instances in which a limit can fail to exist. It can fail to exist because the left- and right-hand limits are not equal, or it can fail to exist because either the left- or right-hand limit fails to exist.

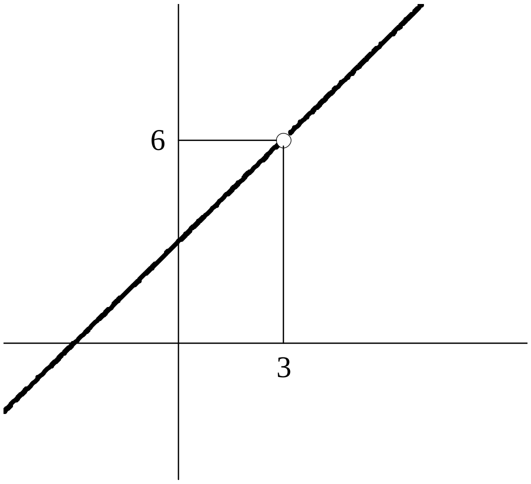
Note: In mathematics we commonly use the symbol $\nexists$ to denote "does not exist" or "fails to exist".

Vertical asymptotes

You may have noticed already that whenever we have a limit that increases or decreases without bound (blows up or down) as $x\to a$, we have a vertical asymptote. In fact, this is the definition of a vertical asymptote.

> The function *f* has a vertical asymptote at $x=a$ if and only if $f(x)\to\pm\infty$ as $x\to a$ from the left or the right.

We must use limits to find vertical asymptotes. A very common error for students to make is assuming that because a function fails to exist at a point, it must have a vertical asymptote there. Consider the function $f(x)=\dfrac{x^2-9}{x-3}$. The graph of *f* is shown below.



f does not exist at $x = 3$ but $\lim\limits_{x \to 3} f(x) = 6$. And clearly, there is no vertical asymptote at $x = 3$. Instead there is just a "hole" in the graph. This is why it is important to show that the limit of $f(x)$ fails to exist as $x \to a$ from the left or right.

Summary

Here are some of the important items to remember from this section:

- $\lim\limits_{x \to a} f(x) = L$ only if $\lim\limits_{x \to a^+} f(x) = \lim\limits_{x \to a^-} f(x) = L$.
- A function does not have to exist at $x = a$ to have a limit as $x \to a$.
- Finding the value of a function at a point, and finding the value of the limit at the same point are <u>not</u> the same thing.

Limit Theorems

Introduction

We discovered in the previous section that we could answer questions about limits if we are given the graph of a function. When given an algebraic expression or a function in algebraic form, all we have been able to thus far is estimate a limit—normally using a table. In this section we will learn methods to find the exact value of a limit when presented with an algebraic expression or function. As in the previous section, we will be addressing only limits as $x \to a$, limits at infinity ($x \to \pm\infty$) will come later.

How to find a limit as x ® a

Before we discuss the technique we actually use to find a limit as $x \to a$, we need to remind ourselves that finding a limit is NOT the same as finding a function value. That being said, the first step in finding the limit of an expression is to "plug" the ***a*** into the expression. The next step depends on what happens as a result of "plugging in ***a***." The table below summarizes the procedure.

Result of replacing the *x*'s with ***a***'s	What the result means	What to do next
a constant, k	the limit of the expression is k	we're finished
an indeterminant result like $\frac{0}{0}$	we are not finished	rationalize the numerator, factor, or use L'Hopital's Rule*
an undefined expression $\frac{k}{0}$	the limit does not exist	for some problems we are finished…for other problems we will need to find left- and right-hand limits to determine the behavior of the expression close to ***a***

*We will learn L'Hopital's Rule later on, after we know how to differentiate.

Example 1

Find $\lim\limits_{x\to 5}(3x+2)$.

$$\lim_{x\to 5}(3x+2) = 17 \qquad (1)$$

Let's be clear about what we are saying here. We are not saying that 3 times 5 plus 2 equals 17. Although this is true, it is not the meaning of the limit statement. All we are saying is that as *x* gets closer and closer to 5, the value of the expression $3x+2$ gets closer and closer to 17. This is an important distinction. It is so important that when you do limit problems, we never want to make it look like we calculated a function value! Do all your arithmetic on a separate piece of scratch paper and simply write the result. All the work you need to show is (1).

Example 2

Find $\lim_{x \to 3} \frac{x-3}{x+5}$.

$$\lim_{x \to 3} \frac{x-3}{x+5} = 0 \qquad (2)$$

Again, all calculations are done out of view. We would not want to write $\lim_{x \to 3} \frac{x-3}{x+5} = \frac{3-3}{3+5} = \frac{0}{8} = 0$ because this makes it look like we're finding the value of the expression at $x = 3$. The problem is asking us to determine what value $\frac{x-3}{x+5}$ approaches as *x* <u>approaches, but never equals</u> 3. All the work you need to show is (2).

Example 3

Find $\lim_{x \to 3} \frac{x^2-9}{x-3}$

$$\begin{aligned}\lim_{x \to 3} \frac{x^2-9}{x-3} &= \lim_{x \to 3} \frac{(x+3)(x-3)}{x-3} \\ &= \lim_{x \to 3}(x+3) \\ &= 6\end{aligned}$$

Notice that all the algebra steps are shown, but once again there is no indication that we ever found the limit by finding the value of any expression.

This might be a good time to address when we can and when we cannot reduce common factors. Consider the function $f(x) = \frac{x^2-9}{x-3}$. We can rewrite *f* as $f(x) = \frac{(x-3)(x+3)}{x-3}$ but we cannot reduce the common factor $x-3$ and say $f(x) = \frac{x^2-9}{x-3} = \frac{(x-3)(x+3)}{x-3} = x+3$. If two functions have different domains, they are different functions. The original *f*, with the $x-3$ in the denominator does not exist at $x = 3$. On the other hand, the function $f(x) = x+3$ does exist at $x = 3$ and thus is a different function.

When we take a limit however, we are allowed to reduce the common factors. This is because when we take the limit, we are letting *x* approach 3 but never letting it equal 3. Since *x* never equals 3, we are not dividing by zero.

Example 4

Find $\lim\limits_{x\to 9}\dfrac{\sqrt{x}-3}{x-9}$.

The result of substituting 9 for *x* is $\dfrac{0}{0}$ which means we are not finished.

We will find the limit in two ways, first by rationalizing the numerator, then by factoring.

$$\begin{aligned}\lim_{x\to 9}\frac{\sqrt{x}-3}{x-9} &= \lim_{x\to a}\frac{\sqrt{x}-3}{x-9}\bullet\frac{\sqrt{x}+3}{\sqrt{x}+3}\\ &= \lim_{x\to 9}\frac{x-9}{(x-9)\left(\sqrt{x}+3\right)}\\ &= \lim_{x\to 9}\frac{1}{\left(\sqrt{x}+3\right)}\\ &= \frac{1}{6}\end{aligned}$$

$$\begin{aligned}\lim_{x\to 9}\frac{\sqrt{x}-3}{x-9} &= \lim_{x\to a}\frac{\sqrt{x}-3}{\left(\sqrt{x}-3\right)\left(\sqrt{x}+3\right)}\\ &= \lim_{x\to 9}\frac{1}{\left(\sqrt{x}+3\right)}\\ &= \frac{1}{6}\end{aligned}$$

Example 5

Find $\lim\limits_{x\to -7}\dfrac{1}{x+7}$

The result of substituting -7 for the *x* is $\dfrac{1}{0}$ which means the limit does not exist. At this point we could just say

$$\lim_{x\to -7}\frac{1}{x+7}\nexists.$$

More often than not, we want to tell the reader how the function behaves as $x\to -7$. To do this, we will find left- and right-hand limits. First, think of a number just to the right of –7, like –6.999. If we substituted –6.999 for x, the numerator would be positive and the denominator would be positive, thus

$$\lim_{x\to -7^+}\frac{1}{x+7}=+\infty$$

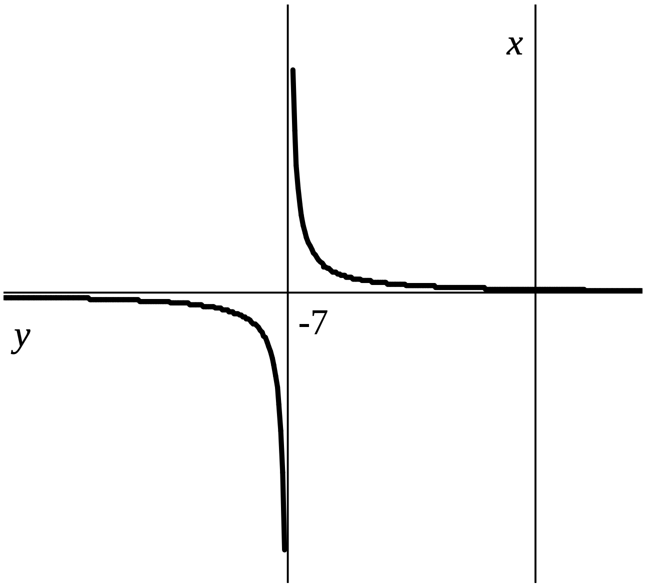
Now think of a number just to the left of –7, like –7.001. If we substitute –7.001 for the *x*, the numerator is positive but the denominator is negative thus

$$\lim_{x\to -7^+} \frac{1}{x+7} = -\infty .$$

A complete solution to this problem should look like this:

$$\lim_{x\to -7} \frac{1}{x+7} \nexists \text{ but } \lim_{x\to -7^+} \frac{1}{x+7} = +\infty \text{ and } \lim_{x\to -7^+} \frac{1}{x+7} = -\infty .$$

Again, all calculations were done on scratch paper to avoid the appearance of finding the value of the expression as opposed to finding the limit. You do not have to include a graph, but the graph of $f(x) = \frac{1}{x+7}$ is shown below to demonstrate the behavior of the function as $x \to -7$.



Limits of piecewise functions

Whenever faced with a piecewise function or a function involving absolute value, we must begin the problem by finding both left- and right-hand limits.

Example 6

Given $f(x) = \begin{cases} \sqrt{x-4} & \text{if } x > 4 \\ 8-2x & \text{if } x < 4 \end{cases}$, find $\lim_{x\to 4} f(x)$.

Note that *f* does not exist at $x = 4$ but remember that it is not necessary for a function to exist at a to have a limit as $x \to a$. To find the limit from the right, we need to use the top piece because we are approaching 4 from values of x which are slightly greater than 4. To find the limit from the left we will use the bottom piece. **The work shown below is exactly how you would present your solution!**

$$f(x)=\begin{cases}\sqrt{x-4} & \text{if } x>4\\ 8-2x & \text{if } x<4\end{cases}$$

$$\text{Since } \lim_{x\to 4^+} f(x)=0 \text{ and } \lim_{x\to 4^-} f(x)=0,\ \lim_{x\to 4} f(x)=0.$$

Example 7

Given $f(x)=\begin{cases}\sqrt{x-3} & \text{if } x>4\\ 8-2x & \text{if } x<4\end{cases}$, find $\lim_{x\to 4} f(x)$.

$$f(x)=\begin{cases}\sqrt{x-3} & \text{if } x>4\\ 8-2x & \text{if } x<4\end{cases}$$

$$\text{Since } \lim_{x\to 4^+} f(x)=1 \text{ but } \lim_{x\to 4^-} f(x)=0,\ \lim_{x\to 4} f(x)\ \nexists.$$

Example 8

Given $f(x)=|x-2|$, find $\lim_{x\to 2} f(x)$.

All functions that involve absolute value are actually piecewise functions! To work with these functions we must first rewrite them as piecewise functions. To do this, first determine where the function "breaks" by setting the expression inside the absolute value equal to zero. For our problem, the function can be rewritten as

$$f(x)=\begin{cases}x-2 & \text{if } x\ge 2\\ 2-x & \text{if } x<2\end{cases}$$

$$\text{Now, since } \lim_{x\to 2^+} f(x)=0 \text{ and } \lim_{x\to 2^-} f(x)=0,\ \lim_{x\to 2} f(x)=0.$$

Limit theorems

Most of the theorems that apply to limits are intuitive in nature and so their proofs will not be shown. We have actually used many of these theorems in the previous examples. We list them here for reference.

$$\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

The limit of a sum is the sum of the limits.

$$\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$$

The limit of a difference is the difference of the limits.

$\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x)$ where c is a constant.

Constants can be "brought outside" the limit statement.

Example: $\lim_{x\to 2} 8x^3 = 8\lim_{x\to 2} x^3$

$$\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \bullet \lim_{x\to a} g(x)$$

The limit of a product is the product of the limits.

Example: $\lim_{x\to 4}[x^2(8x-2)] = \lim_{x\to 4} x^2 \bullet \lim_{x\to 4}(8x-2)$

$\lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ where $\lim_{x\to a} g(x) \neq 0$

The limit of a quotient is the quotient of the limits.

$$\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$$

Example: $\lim_{x\to -9}(7+4x)^3 = \left[\lim_{x\to -9}(7+4x)\right]^3$

This theorem also holds for rational exponents.

$\lim_{x\to a}[c] = c$ where c is a constant.

Example: $\lim_{x\to -3} 8 = 8$

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)$$

Example: $\lim_{x\to 0}\left[\sin 6x\right] = \sin\left(\lim_{x\to 0} 6x\right)$

Limits of trigonometric functions

We will use only two theorems to find limits involving the trigonometric functions.

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

$$\lim_{x\to 0}\frac{1-\cos x}{x}=0$$

The proof of the second makes use of the first…and the proof of the first makes use of a theorem known as the Squeeze Theorem or Sandwich Theorem. The proof of the Squeeze Theorem is "beyond the scope of this course" and so we present it here without proof.

If $f(x)\le g(x)\le h(x)$ for all x in an open interval that contains a

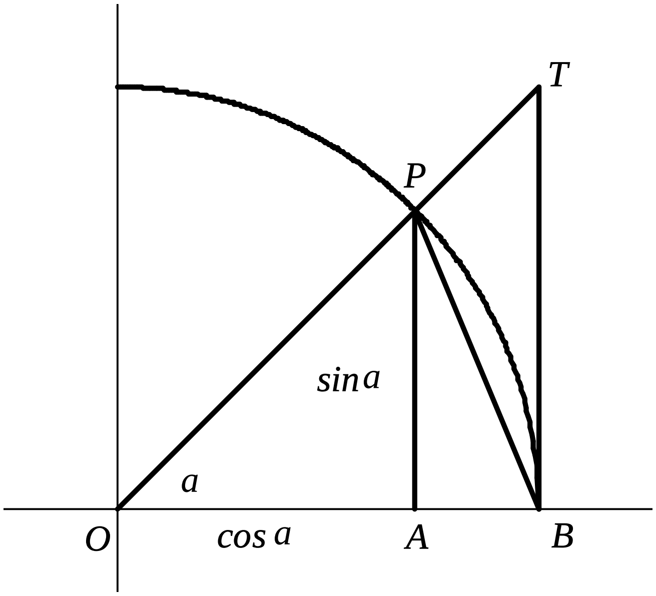
and

$\lim_{x\to a} f(x)=L$ and $\lim_{x\to a} h(x)=L$

then $\lim_{x\to a} g(x)=L.$

The Squeeze Theorem basically tells us that if the value of a function $g(x)$ always lies between the value of two other functions $f(x)$ and $h(x)$, and if *f* and *h* both approach the same number as $x\to a$, then *g* must also approach that same number as $x\to a$.

We will now prove that $\lim_{x\to 0}\frac{\sin x}{x}=1$. Consider the diagram below.



The diagram shows the portion of the unit circle in the first quadrant and thus the distance from *O* to *A* is $\cos a$ and the distance from *A* to *P* is $\sin a$.

Let's first find the coordinates of point *T*—we'll need them to describe several distances on the diagram. The coordinates of *P* are $(\cos a, \sin a)$, thus the equation of the line passing through *P* and the origin is $y = \frac{\sin a}{\cos a} x$. Since the circle is the unit circle, the *x*-coordinate of *T* is 1. Substituting $x = 1$ into the equation for the line yields the *y*-coordinate of *T*, namely $\frac{\sin a}{\cos a}$. Therefore the coordinates of *T* are $\left(1, \frac{\sin a}{\cos a}\right)$.

Here's the relationship that will set up our proof. Notice that the area of triangle *BOP* is less than the area of the sector of the unit circle *BOP* which is in turn less than the area of triangle *BOT*.

The area of triangle *BOP* is $\frac{1}{2}\sin a$ …just one-half the base (1) times the height ($\sin a$).

The area of the sector *BOP* is $\frac{1}{2}a$ …the area of a sector being $\frac{1}{2}r^2 a \;\; (r = 1)$.

The area of triangle *BOT* is $\frac{1}{2}\frac{\sin a}{\cos a}$ …one-half the base (1) times the height (the *y*-coordinate of *T*).

We can now write

$$\frac{1}{2}\sin a < \frac{1}{2}a < \frac{1}{2}\frac{\sin a}{\cos a}$$

Multiplying through by 2 yields

$$\sin a < a < \frac{\sin a}{\cos a}$$

Now divide through by $\sin a$ (which is positive in quadrant one so the inequalities are unchanged).

$$1 < \frac{a}{\sin a} < \frac{1}{\cos a}$$

We will now take the reciprocal of each term. As we do this, we have to reverse the direction of the inequalities.

$$1 > \frac{\sin a}{a} > \cos a$$

Rewriting the inequality as a "less than" statement yields

$$\cos a < \frac{\sin a}{a} < 1$$

Using the procedure outlined in the table at the beginning of this section we see that

$$\lim_{a \to 0} \cos a = 1 \text{ and } \lim_{a \to 0} 1 = 1$$

Therefore, by the Squeeze Theorem,

$$\lim_{a \to 0} \frac{\sin a}{a} = 1$$

$$\boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1}$$

The basic tactic in finding limits that involve the sine function is to make parts of the expression you are working with look like the theorem. Keep in mind that *x* could be anything, so the theorem can be used to state $\lim\limits_{8x \to 0} \dfrac{\sin 8x}{8x} = 1$. Note that as $x \to 0,\ 8x \to 0$ so we can write $\lim\limits_{x \to 0} \dfrac{\sin 8x}{8x} = 1$.

Example 9

Find $\lim\limits_{x \to 0} \dfrac{\sin 7x}{2x}$

$$\begin{aligned}
\lim_{x \to 0} \frac{\sin 7x}{2x} &= \lim_{x \to 0} \frac{\frac{\sin 7x}{7x}}{\frac{2x}{7x}} \\
&= \lim_{x \to 0} \frac{\sin 7x}{7x} \bullet \frac{7x}{2x} \\
&= \lim_{x \to 0} \frac{\sin 7x}{7x} \bullet \lim_{x \to 0} \frac{7x}{2x} \\
&= \frac{7}{2}
\end{aligned}$$

In the second to the last step, since *x* is approaching zero and never equaling zero, we are allowed to reduce the *x*'s in $\dfrac{7x}{2x}$ to 1.

Example 10

Find $\lim\limits_{x \to 0} \dfrac{2x}{\sin 9x}$

$$\begin{aligned}
\lim_{x \to 0} \frac{2x}{\sin 9x} &= \lim_{x \to 0} \frac{\frac{2x}{9x}}{\frac{\sin 9x}{9x}} \\
&= \frac{\lim\limits_{x \to 0} \frac{2x}{9x}}{\lim\limits_{x \to 0} \frac{\sin 9x}{9x}} \\
&= \frac{2}{9}
\end{aligned}$$

Example 11

Find $\lim_{x\to 0}\frac{\sin^2 3x}{18x^2}$

$$\begin{aligned}\lim_{x\to 0}\frac{\sin^2 3x}{18x^2} &= \lim_{x\to 0}\frac{\sin 3x}{3x}\bullet\frac{\sin 3x}{3x}\bullet\frac{1}{2}\\ &= \frac{1}{2}\end{aligned}$$

Now that we have a limit theorem that allows us to work with functions involving the sine function, we need one for cosine.

Consider $\lim_{x\to 0}\frac{1-\cos x}{x}$

$$\begin{aligned}\lim_{x\to 0}\frac{1-\cos x}{x} &= \lim_{x\to 0}\frac{1-\cos x}{x}\bullet\frac{1+\cos x}{1+\cos x}\\ &= \lim_{x\to 0}\frac{1-\cos^2 x}{x(1+\cos x)}\\ &= \lim_{x\to 0}\frac{\sin^2 x}{x(1+\cos x)}\\ &= \lim_{x\to 0}\left[\frac{\sin x}{x}\bullet\frac{\sin x}{1+\cos x}\right]\\ &= 1\bullet 0\\ &= 0\end{aligned}$$

Example 12

Find $\lim_{x\to 0}\frac{1-\cos 4x}{x}$

$$\begin{aligned}\lim_{x\to 0}\frac{1-\cos 4x}{x} &= \lim_{x\to 0}\frac{\frac{1-\cos 4x}{4x}}{\frac{x}{4x}}\\ &= \lim_{x\to 0}\left[\frac{1-\cos 4x}{4x}\bullet\frac{4x}{x}\right]\\ &= 0\bullet 4\\ &= 0\end{aligned}$$

Let's finish up with one last item. The first step you should always take in finding a limit, whether or not it involves a trigonometric function, is to substitute the a for the x and see what happens. For example, finding $\lim_{x\to 0}(\sin 7x)$ does not require the same procedures used in the last few examples. If you substitute 0 in for the *x*, the result is $\sin 0 = 0$ so $\lim_{x \to 0} \left(\sin 7x \right) = 0$. The only time you will need to resort to the trigonometric limit theorems we just discussed is when, after substituting *a* for *x*, you get a $\frac{0}{0}$.

The Formal Definition of Limit

Introduction

The concept of limit lies at the very core of calculus. To understand calculus we must first understand what a limit is…what it means for a function to have a limit. Now, we would expect that such a central concept would have been well developed first, laying a foundation for all that followed. This however, was not the case. Although the invention of calculus occurred around 1690 or so, a formal, rigorous definition of limit did not exist until around 1845! For almost 200 years, mathematicians pushed ahead, extending calculus and its applications without a real foundation for its most central concept. In fact, there were mathematicians (Rolle, Berkley, etc.) who were vehemently opposed to the continued study of calculus until it could be put on rigorous foundations. The basic problem can be found in an expression you've already seen repeatedly the past several years… $\frac{f(x+h)-f(x)}{h}$. Consider the function $f(x)=x^2$. If $f(x)=x^2$, then

$$\begin{aligned}\frac{f(x+h)-f(x)}{h}&=\frac{(x+h)^2-x^2}{h}\\&=\frac{x^2+2xh+h^2-x^2}{h}\\&=\frac{h(2x+h)}{h}\end{aligned}$$

Now, as long as the h is not equal to zero we can simplify the expression to $2x+h$. Here's where the problem reared its ugly head. Mathematicians working with calculus had developed a concept called the "derivative". (We will spend a great deal of time with derivatives very soon!) Now, everyone who was doing calculus "knew" the derivative of x^2 was $2x$ and they knew how to get it. They applied the famous quotient $\frac{f(x+h)-f(x)}{h}$ to x^2. But how to get from $\frac{h(2x+h)}{h}$ to $2x$…that was the problem. If the h is not zero, then you can reduce the expression to $2x+h$. However, to get from $2x+h$ to $2x$, they had to let $h=0$! And they did. Their argument went something like this "…well…you see…up here at the start the h is never zero but in the final step it is!" You can see how this argument, given by some very powerful mathematicians would cause some controversy. Eventually Karl Wierstrauss, extending the work of Louis Cauchy, developed the modern definition of limit. This definition allows us to get around having to explain how the h in the quotient $\frac{f(x+h)-f(x)}{h}$ is at one point never equal to zero but in the end, really is zero. George Berkeley (1685-1753), one of calculus' detractors, referred to the h as a "ghost of a departing quantity".

A verbal definition of limit

Before we move on to a more precise definition of limit, let's state the definition in more informal, verbal terms.

> The limit, as *x* approaches *a*, of a function *f* is equal to *L* if we can make the values of the function as close to *L* as we want by choosing inputs, *x*'s, sufficiently close to but never equal to *a*.

Consider the function $f(x) = 2x+1$. If we say $\lim_{x\to 3} f(x) = 7$, we are saying that we can make the function values as close to 7 as we want by choosing *x*'s close enough to 3. It's like someone saying to us, "I want the value of $f(x) = 2x+1$ to be within .500 units of 7. Can you tell me how close to $x = 3$ I should choose my inputs?" Of course we answer them. The largest function value the person wants is 7.500 and the smallest value is 6.500. Let's set both of these values equal to $2x+1$ and solve for *x*.

$$\begin{array}{lcl} 7.500 = 2x+1 & & 6.500 = 2x+1 \\ 6.500 = 2x & \text{and} & 5.500 = 2x \\ \quad x = 3.250 & & \quad x = 2.750 \end{array}$$

Since 3.250 is .250 units from 3 and 2.750 is .250 units from 3, if we choose inputs that are all within .250 units of $x = 3$, the function values will always be within .500 units of 7.

Now suppose they ask, "Ok, but now I want all my function values to be within .100 units of 7. Can you tell me how close to $x = 3$ I should choose my inputs?" We could easily answer them by repeating the calculation above with function values of 7.100 and 6.900.

In fact, we can always find an appropriate range for the inputs that will keep the function value as close to 7 as we want. This is exactly what $\lim_{x\to 3}(2x+1) = 7$ means.

In order to make the rest of the discussion clearer, we will now introduce some very important notation. In the above example, when the person said they wanted all the function values to be within .500 units of 7, they wanted all the function values to be in the interval [6.500, 7.500]. The .500 is a distance…a distance from the function value 7. We call such a distance e (epsilon). When we finished our calculations and told them they should stay within .025 units of 3. All in inputs must be in the interval [2.750,3.250].]. This distance, .025, from 3 where we must choose all inputs is called a d (delta).

The formal definition of limit

In the formal definition of limit, inequalities are used to describe the distances epsilon and delta. Let's consider the simple inequality $|x-3| < 4$. Using the definition of absolute value, we can say

$$x-3 < 4 \text{ and } x-3 > -4$$
$$x < 7 \text{ and } x > -1$$

The solution to this inequality can be written $-1 < x < 7$. Notice that the midpoint of the interval $(-1,7)$ is 3. This distance from -1 to 3 is 4 and the distance from 7 to 3 is 4. In other words, the inequality $|x-3| < 4$ describes all the x's that are within 4 units of 3. Similarly, the inequality $|x-5| < 2$ describes all the x's that are within 2 units of 5.

In our original problem, the person wanted all the function values to be within .500 units of 7. A very efficient way to state this to use the inequality $|f(x)-7| < .500$. This inequality describes all the function values that are within .500 units of 7—just what they asked for!

When we said that all of the inputs must be within .025 units of 3, we could have used the inequality $|x-3| < .025$. This inequality describes all the x's that are within .025 units of 3.

Remember, epsilon is the distance from L within which we want all of our function values. This relationship can be precisely and elegantly stated with the inequality $|f(x)-L| < e$. This inequality clearly describes all the function values that are within e units of L.

Delta was the distance from a within which we had to choose all of our inputs. Again, this relationship can be stated with $|x-a| < d$. This inequality describes all the x's that are within d units of a.

We're now ready for a formal definition of limit.

Definition of Limit

$\lim_{x \to a} f(x) = L$ is true if

for any $e > 0$, there exists a $d > 0$ such that whenever $|x-a| < d$ then $|f(x)-L| < e$.

We can clean it up further by using several common mathematical symbols. "$\exists$" is mathematics shorthand for "there exists". "$\forall$" means "for any" or "for all". The symbol "$\to$" is used in place of the words "implies" or "leads to" and the symbol "$\ni$" means "such that"

Definition of Limit

$\lim_{x \to a} f(x) = L$ is true if $\forall\ e > 0\ \exists\ d > 0\ \ni$

whenever $|x-a| < d \to |f(x)-L| < e$.

The delta and epsilon are both distances and cannot be negative. This is why the definition contains the "$e > 0$" and "$d > 0$". If we can find a delta for any epsilon, the limit statement is true.

It is hard to emphasize enough the importance of this definition. Its invention by Karl Weierstrass (1815-1897) finally put calculus on solid, rigorous foundations. Without it, the entire subject is a house of cards. It is the central tenant of calculus.

Working with the definition

There are two types of problems we face which directly involve the definition of limit. In one problem, we are given a limit statement we are told is true and an e . Our task is to find an appropriate d . The second type of problem is proving limit statements. We are given a limit statement and asked to prove it.

Example 1

For $e = .010$, find an appropriate d such that $\lim_{x\to 3}(2x-1) = 5$ is true.

To find the delta, we will apply the definition of limit.

For $e = .01$ we need to find a $d > 0$ such that whenever $|x-3| < d \to |(2x-1)-5| < .010$

We proceed by working with the inequality $|(2x-1)-5| < .010$.

$$|(2x-1)-5| < .010$$

$$|2x-6| < .010$$

$$2|x-3| < .010$$

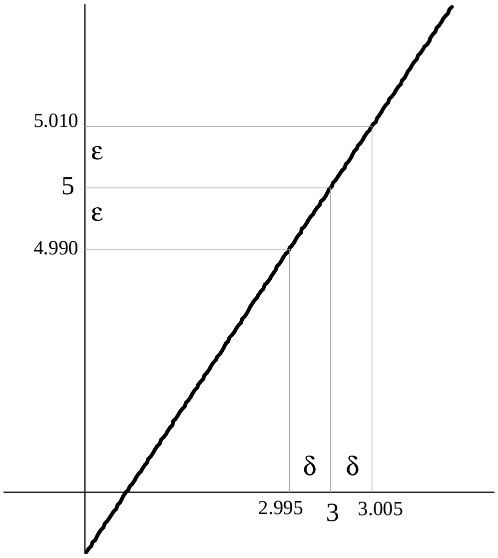
$$|x-3| < .005$$

Consider now what we have. We wanted $|x-3| < d$ and we now know that $|x-3| < .005$!

We therefore choose $d \le .005$

So what exactly have we done here? We've shown that if we stay within .005 units of 3, all the function values will be within .010 of 5.

You might be wondering why we choose $d \le .005$ and not $d = .005$. We use "≤" because if a delta of .005 works, then any delta smaller than .005 will work. The diagram below shows both the delta and epsilon.



Example 2

For $e = .020$, find an appropriate d such that $\lim\limits_{x \to -2} (5x+1) = -9$ is true.

For $e = .020$ we need to find a $d > 0$ such that whenever $|x+2| < d \ \rightarrow \ |(5x+1)+9| < .020$.

$$|x+2| < d \ \rightarrow \ |(5x+1)+9| < .020$$

$$|5x+10| < .020$$

$$5|x+2| < .020$$

$$|x+2| < .004$$

Therefore we choose $d \le .004$.

Thus far, all the functions we have used have been linear. In the next example we will use a non-linear function. Before we do so, we need to know just one more thing about delta. By convention, we never give a delta that is greater than one. It's a matter of keeping things tidy. Even if we are given a huge epsilon which algebraically generates a delta larger than one, we would choose delta to be less than or equal to one, not something larger than one.

Example 3

For $e = .100$, find an appropriate d such that $\lim\limits_{x \to 3} (x^2 - 2x + 1) = 4$ is true.

For $e = .100$, we need to find a $d > 0$ such that whenever $|x-3| < d \ \rightarrow \ |(x^2 - 2x + 1) - 4| < .100$.

$$|x-3| < d \ \rightarrow \ |(x^2 - 2x + 1) - 4| < .100$$

$$|x^2 - 2x - 3| < .100$$

$$|(x-3)(x+1)| < .100$$

$$|x-3||x+1| < .100$$

$$|x-3| < \frac{.100}{|x+1|}$$

Notice that what we would normally choose to be delta at this point is dependent on *x*. We cannot choose such a delta. However, we know that we will never give a delta greater than one so no matter what *x* we use an input, it will always be in the interval (2,4)—the interval one unit to the left and right of $x = 3$. We proceed by considering the interval (2,4).

Consider (2,4).

$$\text{If } x = 2 \rightarrow \frac{.100}{|x+1|} = \frac{.100}{3}$$

$$\text{If } x = 4 \rightarrow \frac{.100}{|x+1|} = \frac{.100}{5}$$

We therefore choose $d \le \frac{.100}{5}$. (Because it is the smaller of the two choices.)

We will now move on to the second type of delta-epsilon problem, proving limit statements. We do them in much the same way we did problems in which we found an appropriate delta for a given epsilon. The basic difference is that we must show that there is a delta for <u>any</u> epsilon, not just a particular epsilon.

Example 4

Prove $\lim_{x\to 6}(3x+2)=20$

For $\forall e>0$ we need to show that there exists a $d>0 \ni$ whenever

$$|x-6|<d \rightarrow |(3x+2)-20|<e$$

$$|3x-18|<e$$

$$3|x-6|<e$$

$$|x-6|<\frac{e}{3}$$

Therefore, choose $d=\min\left\{1,\frac{e}{3}\right\}$

" choose $d=\min\left\{1,\frac{e}{3}\right\}$ " means that, depending on what e is, we always choose the smaller of 1 or $\frac{e}{3}$.

If someone were to give us an epsilon of 15, then $\frac{e}{3}=5$ but we do not want $d>1$ so if $e=15$, we would respond with $d=1$.

Example 5

Prove $\lim_{x\to -1}(x^2-5x+1)=7$

For $\forall e>0$ we need to show that there exists a $d>0 \ni$ whenever

$$|x+1|<d \rightarrow |(x^2-5x+1)-7|<e$$

$$|x^2-5x-6|<e$$

$$|x+1||x-6|<e$$

$$|x+1|<\frac{e}{|x-6|}$$

Consider the interval $(-2,0)$.

If $x=-2 \rightarrow \frac{e}{|x-6|}=\frac{e}{8}$

If $x=0 \rightarrow \frac{e}{|x-6|}=\frac{e}{6}$

Therefore choose $d=\min\left\{1,\frac{e}{8}\right\}$

Continuity

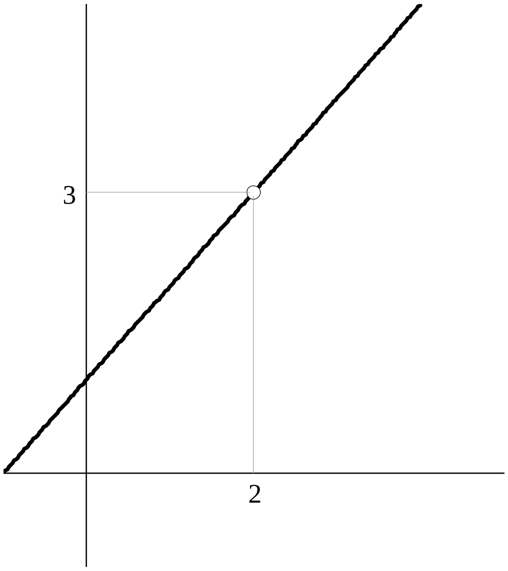
Introduction

The meaning of the term *continuity* in mathematics has much the same connotation as it does in everyday usage. It implies a smoothly changing phenomenon—something that does not change abruptly or stop and then start again. Calculus was invented in part to answer questions about continuous phenomena so it makes sense that continuity is a very important concept to us. Many of the theorems we encounter will only hold for functions that are *continuous* on a particular interval.

We will study two types of continuity…continuity at a number and continuity on an interval.

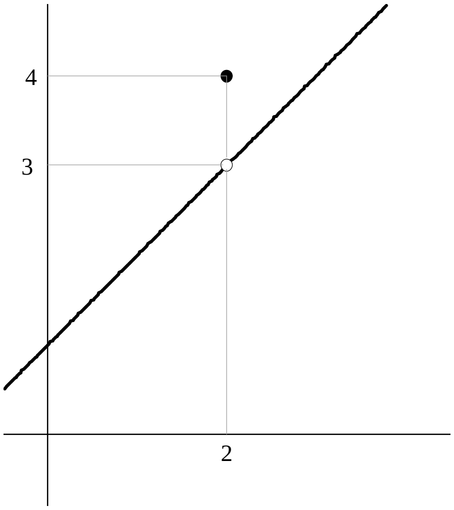
Continuity at a number

To establish a definition of continuity at a number we will examine several different functions. We will study each function at a particular number by finding a function value (if there is one) and the limit of the function at the number (if there is a limit). First, consider the function $f(x)=\dfrac{x^2-x-2}{x-2}$ at $x=2$. The graph of *f* is given below.



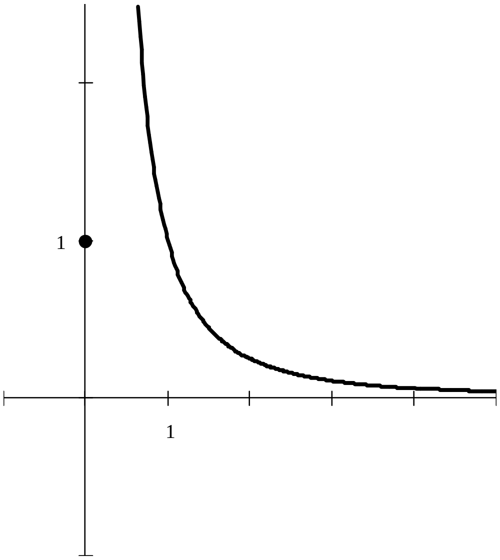
Note that $f(x)\nexists$ at $x=2$ but $\lim\limits_{x\to 2} f(x)=3$. Here we have a function that has a limit but no function value at the number in question. We conclude that *f* is not continuous at $x=2$.

Let's change the function just a little and perform the same analysis. Let $f(x)=\begin{cases}\dfrac{x^2-x-2}{x-2} & \text{if } x\neq 2\\ 4 & \text{if } x=2\end{cases}$. The graph of *f* is shown below.



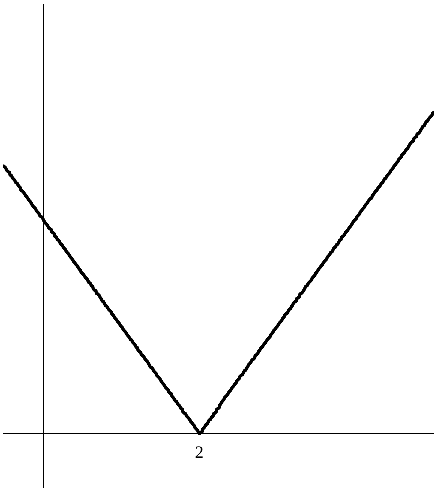
Now we can see that $f(2)=4$ but $\lim_{x\to 2} f(x)=3$. Even though *f* has both a function value and a limit at $x=2$, we still say that *f* is not continuous at $x=2$.

Consider the function $f(x)=\begin{cases}\dfrac{1}{x^2} & \text{if } x>0\\ 1 & \text{if } x=0\end{cases}$. The graph of *f* is given below.



Clearly $f(0)=1$ but $\lim_{x\to 0} f(x)\not\exists$. (As we approach zero from the right the function values increase without bound. We cannot approach zero from the left because *f* is not defined for any $x<0$.) Unlike the previous two functions, here we have a situation where *f* has a function value at the number in question ($x = 0$) but has no limit there. We conclude that *f* is not continuous at $x = 0$.

Let's consider one more function, $f(x) = |x-2|$ at $x = 2$.



For this function, $f(2) = 0$ and $\lim_{x \to 2} f(x) = 0$. This is the only function we have considered in which both the function value and the limit at the number in question were equal. We say that *f* is continuous at $x = 2$.

Now, it is never a good idea to generalize from a handful of examples but it would appear that for a function to be continuous at a number, the function must exist at the number, the function must have a limit at the number and both the function value and limit must be equal. In fact, this is the definition of continuity at a number.

Definition of continuity of a function at $x = a$

A function *f* is continuous at $x = a$ if and only if

i. $f(a)$ exists

ii. $\lim_{x \to a} f(x)$ exists

iii. $f(a) = \lim_{x \to a} f(x)$

Example 1

Determine if $f(x) = \dfrac{x^2 - 25}{x - 5}$ is continuous at $x = 5$.

<u>Continuity test at $x = 5$</u>

Since $f(5) \nexists$, f is not continuous at $x = 5$.

Notice that continuity tests can be quite simple—especially if $f(a)$ fails to exist!

Example 2

Determine if $f(x)=\begin{cases}\dfrac{x^2-25}{x-5} & \text{if } x\neq 5\\ 7 & \text{if } x=5\end{cases}$ is continuous at $x=5$.

<u>Continuity test at $x = 5$</u>

$f(5)=7$ but $\lim\limits_{x\to 5} f(x)=10$. Since $f(5)\neq\lim\limits_{x\to 5} f(x)$ *f* is not continuous at $x=5$.

Please note a common student error. The statement "$f(5)=7$ but $\lim\limits_{x\to 5} f(x)=10$ therefore *f* is not continuous at $x=5$" does not tell the reader explicitly that the student knows why *f* is not continuous. Please make sure you always use the definition explicitly and tell the reader exactly why a function is not continuous. In this case we need to clearly state "Since $f(5)\neq\lim\limits_{x\to 5} f(x)$, *f* is not continuous at $x=5$." I know it sounds picky but that's just the way we have to do it.

Example 3

Determine if $f(x)=\begin{cases}3x-4 & \text{if } x\leq 2\\ x^2+1 & \text{if } x>2\end{cases}$ is continuous at $x=2$.

<u>Continuity test at $x = 2$</u>

$$f(2)=2$$

$$\lim_{x\to 2^+} f(x)=5 \text{ but } \lim_{x\to 2^-} f(x)=2 \text{ therefore } \lim_{x\to 2} f(x)\ \nexists$$

Since $\lim\limits_{x\to 2} f(x)\ \nexists$, *f* is not continuous at $x=2$.

Example 4

If $g(x)=\begin{cases}\dfrac{x^2-25}{x-5} & \text{if } x\neq 5\\ b & \text{if } x=5\end{cases}$, for what value of *b* will *g* be continuous at $x=5$?

<u>Continuity test at x = 5</u>

$$g(5)=b$$

To find $\lim\limits_{x\to 5} g(x)$, we need only consider the "top" piece.

$$\lim_{x\to 5} g(x)=10$$

For *g* to be continuous at $x=5$, $g(5)=\lim\limits_{x\to 5} g(x)$, therefore

$$b=10$$

Removable vs. essential discontinuities

Not all discontinuities are created equal. Some of them can be "removed" by redefining the original function while others are "essential" to the function and cannot be removed.

A discontinuity at $x = a$ is called ***removable*** if $\lim\limits_{x \to a} f(x)$ exists.

A discontinuity at $x = a$ is called ***essential*** if $\lim\limits_{x \to a} f(x)$ does not exist.

Example 5

Determine if $f(x) = \dfrac{x^2 - 4}{x - 2}$ is continuous at $x = 2$. If discontinuous, determine if the discontinuity is removable or essential. If removable, redefine f so that it is continuous at $x = 2$.

Continuity test at $x = 2$

Since $f(2) \nexists$, f is discontinuous at $x = 2$.

Removable or essential

Since $\lim\limits_{x \to 2} f(x) = 4$, the limit exists and therefore the discontinuity is removable.

Redefining f

$$f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & \text{if } x \neq 2 \\ 4 & \text{if } x = 2 \end{cases}$$

As you can see, if a discontinuity is removable, you remove it by redefining f so that the value of the function at the discontinuity is equal to the limit of the function at the discontinuity.

Continuity on an interval

Although this is a topic that can become quite detailed, we will make use of several theorems to make our work easier.

Polynomial functions are continuous everywhere.

Rational functions are continuous everywhere in their domains.

Functions of the form $f(x) = \sqrt[n]{x}$ are continuous everywhere if n is odd and continuous on $[0, \infty)$ if n is even.

Example 6

Determine if $f(x)=\dfrac{3}{x-4}$ is continuous on the following intervals: $(3,5)$, $(-1,3)$, $[4,9)$ and $(4,9)$.

f is not continuous on (3,5) because $f(4)$ does not exist.

f is continuous on $(-1,3)$ because *f* exists $\exists\ \forall\ x\in(-1,3)$.

f is not continuous on [4,9) because $f(4)$ does not exist.

f is continuous on (4,9) because *f* exists $\exists\ \forall\ x\in(4,9)$.

The Intermediate Value Theorem

The Intermediate Value Theorem belongs to a class of theorems known as *existence theorems*. We will encounter several very important existence theorems this year. The Intermediate Value Theorem is introduced now because it is based only on the continuity of a function.

The Intermediate Value Theorem

If *f* is continuous on the closed interval $[a,b]$ then
$\forall\, k\in\big(f(a),f(b)\big)\ \exists\ c\in(a,b)\ni f(c)=k.$

Let's examine the theorem by "translating" it into a more readable form. Since *k* is in the interval $\big(f(a),f(b)\big)$, *k* is a function value. Since *c* is in (a,b), *c* is clearly an *x*-value. The theorem basically tells us that if a function moves smoothly from one function value to another, we can find any function value in between $f(a)$ and $f(b)$. The theorem guarantees us an *x*-value which will generate any function value we want between $f(a)$ and $f(b)$. Let's look at an example. Consider the function $f(x)=x^2-3$ on $[0,5]$. Since *f* is continuous on $[0,5]$, the Intermediate Value Theorem holds. We also know that $f(0)=-3$ and $f(5)=22$. Since *f* moves smoothly between -3 and 22, we can find any function value in between -3 and 22. The theorem guarantees us that for any function value *k* in $(-3,22)$ there exists an *x*-value (which we will call *c*) in (0,5) such that $f(c)=k$. Take the function value 1. Certainly $1\in(-3,22)$ and so the theorem tells us that there must exist a particular *x*-value (which the theorem calls *c*) which will yield a function value of 1. To find *c*, we set the function equal to 1:

$$x^2-3=1$$
$$x^2=4$$
$$x=2 \text{ or } x=-2$$

Since -2 is not in (0,5), the *c* that theorem guaranteed us must be $c=2$.

The Intermediate Value Theorem is used often to show that a particular function has zeros on a certain interval.

Example 7

Show that $f(x) = x^3$ has at least one zero between $x = -2$ and $x = 5$ without actually finding a zero.

$$f \text{ is continuous on } [-2, 5] \text{ and } f(-2) = -8 \text{ and } f(5) = 125.$$
$$\text{Since } f(-2) < 0 \text{ and } f(5) > 0 \text{ then } f(x) = 0 \text{ for some } x \text{ between } x = -2 \text{ and } x = 5.$$

The Intermediate Value Theorem is also used in the process of proving other theorems. You will often see theorems being proven in which one of the steps will contain the statement, "…thus, by the Intermediate Value Theorem…"

Limits at Infinity

Introduction

We now turn our attention to another type of limit, limits at infinity. The basic question we will be asking is, "How does this function behave as the inputs increase or decrease without bound?" There will be quite a bit of talk about infinity in this section so let's begin by clearing up a common misconception. Infinity is not a number. It's more like a place where numbers go when they increase or decrease without bound. So, when you were a child and argued with a friend or sibling and got to the point where you said, "Infinity plus one!" you really weren't making much sense. The term *infinity* is used to describe behavior—it's where we allow inputs to go…it's where the value of an expression or function goes.

The technique described in this section is not the one normally presented in textbooks. We will be relying on our own common sense, not algebraic techniques.

Relative size

Suppose you just won one million dollars in the lottery. To show how generous you are you give one dollar to your best friend. Well, your best friend may no longer be your best friend…after all, you've only given them one one-millionth of your winnings… $\frac{1}{10^6}$ of your money. And yes, you're being pretty stingy. A few weeks later you win one trillion dollars in the World Powerball Lottery. You decide to be a little more generous with your friend and give them one million dollars. Are you really being any more generous? No. You see, if you give away one million of your one trillion you've once again given away only $\frac{1}{10^6}$ of your money…although your friend may be a little happier, it's only because they probably do not understand the relative sizes of large numbers.

Let's examine the expression x^2+2x. The table below shows some values of $x, 2x$ and x^2.

x	x^2	$2x$	$2x$ as a percent of x^2
10	100	20	20%
20	400	40	10%
50	2500	100	4%
100	10,000	200	2%
1,000	1,000,000	2,000	0.2%
100,000	10,000,000,000	200,000	0.002%
1,000,000	1,000,000,000,000	2,000,000	0.0002%

As x continues to get larger, $2x$ has less and less of an impact on the relative size of x^2+2x. In fact, if we let x increase without bound ($x \to \infty$) the $2x$ eventually has no real impact at all and can be ignored when considering the size of x^2+2x as $x \to \infty$. This same line of thinking holds if we consider an expression like x^3-x^2. As x increases or decreases without bound, the x^2 loses its ability to impact the size of $x^3 - x^2$. In other words, only the term of highest degree in an expression is of any importance to us when taking limits at infinity. This concept will be central to finding limits as $x \to \pm\infty$.

Let us now consider the fraction $\frac{1}{x}$. We are interested in $\lim_{x\to\infty}\frac{1}{x}$. If we let *x* increase without bound, the denominator gets larger and larger while the numerator remains 1. If the denominator of a fraction gets larger while the numerator remains unchanged, the value of the fraction gets smaller and smaller. We say $\lim_{x\to\infty}\frac{1}{x} = 0$. Keep in mind we are not saying that $\frac{1}{x} = 0$ at any time. All we are saying is that as *x* increases without bound, the value of $\frac{1}{x}$ approaches zero.

$$\boxed{\lim_{x\to\pm\infty}\frac{1}{x} = 0}$$

Most of the problems we will encounter for now will involve rational expressions. To find the limit of a rational expression as $x \to \pm\infty$ we begin by comparing the degree of the numerator and denominator.

Let's first examine a limit in which the degree of the numerator is the same as the degree of the denominator…something like $\lim_{x\to\infty}\frac{3x-7}{5x+3}$. In the numerator we can "ignore" the "-7" because as $x \to \infty$ the "-7" will not impact the value of $3x-7$. In the denominator we can "ignore" the "+3" because as $x \to \infty$ it will not impact the value of $5x+3$. This means the $\lim_{x\to\infty}\frac{3x-7}{5x+3}$ is equivalent to $\lim_{x\to\infty}\frac{3x}{5x}$. Now, it won't matter how large the *x* gets, we still have an *x* in both the numerator and denominator which can be reduced to 1. If we reduce the *x*'s, we find $\lim_{x\to\infty}\frac{3x-7}{5x+3} = \frac{3}{5}$.

What happens if the degree of the numerator is less than the degree of the denominator? Let's look at $\lim_{x\to\infty}\frac{x-4}{x^2+6}$. "Ignoring" the –4 and the +6 leaves us with $\lim_{x\to\infty}\frac{x}{x^2}$. Reducing the *x* in the numerator with one in the denominator results in $\lim_{x\to\infty}\frac{1}{x}$ which we know is zero. Therefore $\lim_{x\to\infty}\frac{x-4}{x^2+6} = 0$.

Finally, let's look at a limit in which the degree of the numerator is greater than the degree of the denominator. Consider $\lim_{x\to\infty}\frac{x^3-5x^2+x-1}{x^2+6x+8}$. The highest degreed term in the numerator is the x^3 so we will ignore the "$-5x^2+x-1$". In the denominator, the highest degreed term is the x^2 so the "$+6x+8$" will be ignored. This leaves us with the limit, $\lim_{x\to\infty}\frac{x^3}{x^2}$. Reducing an x^2 from the numerator and denominator yields $\lim_{x\to\infty} x$. Clearly, as $x \to \infty$, $x \to \infty$! This means that $\lim_{x\to\infty}\frac{x^3-5x^2+x-1}{x^2+6x+8} = \infty$. To say a limit "$= \infty$" means that the limit does not exist. (We are just indicating how the limit fails to exist.)

The table below summarizes the technique we use to find a limit as $x \to \pm\infty$.

Relative degree of numerator (^{0}N) and denominator (^{0}D)	**Limit**
$^{0}N < {}^{0}D$	0
$^{0}N > {}^{0}D$	$\pm\infty$ (limit does not exist)
$^{0}N = {}^{0}D$	quotient of coefficient of highest degreed term in numerator and denominator

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Example 1

Find $\lim\limits_{x\to-\infty} \frac{x^2-x}{x^3+8}$.

Since the degree of the numerator is less than the degree of the denominator,

$$\lim_{x\to-\infty} \frac{x^2-x}{x^3+8} = 0$$

In the case where the degree of the numerator is less than the degree of the denominator, it does not matter whether $x \to \infty$ or $x \to -\infty$, the limit is always zero.

Example 2

Find $\lim\limits_{x\to\infty} \frac{x+3}{\sqrt{5x^2-7x}}$

The degree of the numerator and denominator are both 1 so we will <u>carefully</u> divide the coefficients of the highest degreed terms.

$$\lim_{x\to\infty} \frac{x+3}{\sqrt{5x^2-7x}} = \frac{1}{\sqrt{5}}$$

Example 3

Find $\lim\limits_{x\to-\infty} \frac{x+3}{\sqrt{5x^2-7x}}$

The degree of the numerator and denominator are both 1 so we will again <u>carefully</u> divide the coefficients of the highest degreed terms. Note that in the numerator, you will have –1B (where B denotes a really huge number) but in the denominator you will have $\sqrt{5}$ B because when the large negative number is squared, it becomes positive. $\sqrt{5(-B)^2} = \sqrt{5}\sqrt{B^2} = \sqrt{5}\,B$. The really huge number in the numerator and denominator now reduce to one which yields

$$\lim_{x\to-\infty} \frac{x+3}{\sqrt{5x^2-7x}} = -\frac{1}{\sqrt{5}}$$

This example illustrates an important point. We must be extra careful when the degree of the numerator and denominator are equal and the expression involves an even radical.

Example 4

Find $\lim\limits_{x\to\infty}\frac{5x^2-7x-1}{90x+5}$.

Since the degree of the numerator is greater than the degree of the denominator, the limit does not exist. To determine if the expression increases or decreases without bound, pretend we are substituting a huge number in for the *x* in the highest degreed terms. The process going on in our head (but never on our paper) goes something like this: $\frac{5(B^2)}{90B}=\frac{5B}{90}=\infty$.

$$\lim_{x\to\infty}\frac{5x^2-7x-1}{90x+5}=\infty$$

Example 5

Find $\lim\limits_{x\to-\infty}\frac{5x^2-7x-1}{90x+5}$.

This problem is the same as Example 4 but we are letting $x\to-\infty$ instead of $+\infty$. Since the degree of the numerator is greater than the degree of the denominator, the limit does not exist. To determine if the expression increases or decreases without bound, we will again pretend we are substituting a huge number in for the *x* in the highest degreed terms. The process going on in our head goes something like this: $\frac{5(-B)^2}{90(-B)}=\frac{5B^2}{-90B}=\frac{5B}{-90}=-\infty$.

$$\lim_{x\to-\infty}\frac{5x^2-7x-1}{90x+5}=-\infty$$

In general, limits at infinity involve a lot of common sense. Just be careful with your negatives—especially with limits where $x\to-\infty$.