Functions and their Graphs

Functions

All of the functions you will see in this course will be real-valued functions in a single variable. A function is real-valued if the input and output are real numbers as opposed to complex numbers. We will not work with any complex numbers in this course. In precalculus, when you solved the equation $x^2+1=0$, you determined that the solutions were $x=i$ or $x=-i$. In this course, the equation $x^2+1=0$ has no solutions. Can calculus be done with complex numbers? Yes—but it is an entirely different course (one that you should definitely take in the future!) All of our functions will also be functions in a single variable. Nearly all the functions you have ever seen have been in a single variable. Algebraically, functions in a single variable look like $f(x)=x^3$ or $h(x)=\sin x$. Mathematically we can operate with functions in as many variables as we like. A function in two variables would look like $f(x,y)=x^2+y^2$.

There are many ways to define a function. Pick up ten different mathematics texts and you will likely see ten different definitions—all saying the same thing but all phrased in different terms. In general, a function pairs members of an input set with a member or members of an output set.

Here are several definitions of a **function**:

- A function *f* is a rule that assigns to each element *x* in a set A exactly one element, called $f(x)$, in a set B. (Stewart)

- A function from a set D to a set R is a rule that assigns a unique element in R to each element in D. (Finney)

- A function *f* from a set D to a set E is a correspondence that assigns to each element *x* of the set D exactly one element *y* of the set E. (Swokowski)

- A function a is a correspondence from a set A to a set B that associates with each element *a* of A a unique element *b* of B. We denote this correspondence $a(a)=b$ and we call *b* the image of *a* under a. (Larsen)

As you can see, functions are usually defined in terms of sets. Perhaps one of the clearest definitions is the following:

- A function is a set of ordered pairs in which no two distinct ordered pairs have the same first element. (Leithold)

Any set of ordered pairs is called a **relation**. Those sets of ordered pairs which meet any of the definitions listed above are functions. All functions are relations but not all relations are functions.

Throughout the course you will see functions presented in a variety of ways. Functions can be given algebraically, graphically, verbally or in table form.

In terms of notation, there are several methods to express a function algebraically. Suppose we have a function that adds three to an input and then squares the sum. This function will most often be written $f(x)=(x+3)^2$. Another method you may have seen on occasion is $f: x \to (x+3)^2$. The second technique makes a clearer distinction between the name of the function *f*, the input *x*, and the value of the function $(x+3)^2$. More formally, this relationship is written $\{(x, f(x)) \mid f(x)=(x+3)^2\}$.

Domain

The domain of a relation or function is the set of all allowable inputs. Some functions have specific domains such as the natural logarithmic function. If $f(x)=\ln x$, the domain is $(0,\infty)$ because we cannot take the natural logarithm of zero or a negative number—at least not in this course. For most other functions we encounter, we can find the domain by looking for variables in denominators and/or variables under even radicals. Finding the domain of a function is a straightforward procedure! Consider the function $g(x)=\dfrac{1}{x-5}$. Since the denominator cannot be zero, $x \neq 5$, so the domain of g is $(-\infty,5)\cup(5,\infty)$. The function $h(x)=\sqrt{x+8}$, has a domain $[-8,\infty)$ because in order for *h* to exist, $x+8 \geq 0$.

In the following example, we will demonstrate the proper way to show how the domain of a function is found. The days of just looking at a function and writing the domain are over. All solutions require justification, nearly all solution will require words…your solutions should read like paragraphs. This is something we will work on throughout the year!

This example also requires a sign chart—you should already be familiar with how to set them up. The solution below illustrates exactly how you should present it! Look at it carefully!

Example 1

Find the domain of $f(x)=\sqrt{x^2-x-6}$.

f exists when $x^2-x-6 \geq 0$

Now, $x^2-x-6=(x-3)(x+2)$ so $f(x)=0$ when $x=3$ or $x=-2$.

	x^2-x-6
$(-\infty,-2)$	+
$x=-2$	0
$(-2,3)$	–
$x=3$	0
$(3,\infty)$	+

From the chart above, the domain of f is $(-\infty,-2]\cup[3,\infty)$.

Example 2

Find the domain of $p(x)=\dfrac{x+5}{x^2+5x+6}$.

$$p \text{ will not exist when } x^2+5x+6=0.$$

$$x^2+5x+6=0 \text{ when } (x+3)(x+2)=0$$
$$x=-3 \text{ or } x=-2$$
$$\therefore \text{the domain of } p \text{ is } (-\infty,-3)\cup(-3,-2)\cup(-2,\infty).$$

Notice two things about these examples: (1) the domains are given in interval notation and (2) when we deal with even radicals, we normally phrase our process in terms of when the function will exist but when we deal with denominators which cannot be zero, we phrase our process in terms of which inputs (values of *x*) we will eliminate.

In Example 1 we began our solution with "*f* exists when…" but in Example 2 we began with "*p* will not exist when…".

In Example 2, if we began with "*p exists* when…" we would have had to say "…when $x^2+5x+6\neq 0$." The problem is that now many students will write

$$x^2+5x+6\neq 0$$
$$(x+2)(x+3)\neq 0$$
$$x\neq -2 \text{ or } x\neq -3$$

This may look fine but it is not proper mathematics. The steps used to solve a quadratic equation are valid for equations—statements of equality, not statements of "not equals". To avoid this predicament, we state solutions in terms which will allow us to work with "equals" instead of "not equals".

Range

The range of a function is the set of all outputs. Why not "allowable" outputs? Because there is no such thing! Outputs are not "allowable"—they are what they are. Range is generally more difficult to determine than domain. As we move through the course, we will learn more and more sophisticated techniques to find the range of a function. For now, we are somewhat restricted. You can think of the range of a function as the "shadow" (the projection) of the function or relation on the *y*-axis. Until we learn other techniques to find range, we will primarily face two types of range problems. We will need to be able to find the range of functions such as $f(x)=\sqrt{x-4}$ and functions such as $g(x)=\dfrac{x^2-2x-15}{x-5}$. The first function involves an even radical. Because we use the $\sqrt{\ }$ symbol to denote the primary root, its value must always be greater than or equal to zero—so the range of *f* is $[0,\infty)$. g is an example of a <u>rational, factorable function with a common term</u> in the denominator and numerator. A rational function is a function of the form $R(x)=\dfrac{p(x)}{q(x)}$ where both *p* and *q* are polynomials. Note that *g* can be written $g(x)=\dfrac{(x-5)(x+3)}{x-5}$. Since we cannot divide by zero, $x\neq 5$.

Now *imagine* the $x-5$ term disappearing from the numerator and denominator. Now, if $x=5$, the value of the "imagined" function would be 8. Since *x* can never equal 5, $g(x)$ can never equal 8. The range of *g* is $(-\infty,8)\cup(8,\infty)$.

Example 3

Find the range of $f(x)=\dfrac{x^2-9}{x-3}$.

Since $f(x)=\dfrac{x^2-9}{x-3}=\dfrac{(x-3)(x+3)}{x-3}$ and since *f* does not exist when $x=3$, $f(x)$ can never equal 6,

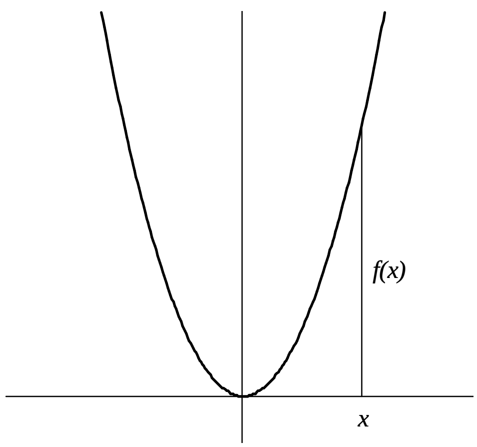
$\therefore$ the range of *f* is $(-\infty,6)\cup(6,\infty)$.

[Please note the presentation of the solution!]

Graphs of functions

We will spend a great deal of time in this course graphing functions. In fact, one of the most important and fascinating areas of calculus is its application to sketching curves. For now there are a few essential ideas we need to know and remember—and the first has to do with labeling. An arbitrary point on a curve can be labeled in one of three ways. Consider the function $f(x)=x^3$. Any point on this curve can be labeled (x,y), (x,x^3) or $(y^{1/3},y)$. Knowing how to label arbitrary points in this manner allows us to work problems in terms of *x* and *y*, in terms of just *x*, or in terms of just *y*.

Another important item to remember is that function values represent distances on the graph of a function. Consider the graph of the function $f(x)=x^2$.



The distance from the origin to the dashed line can be called *x*. The distance from the intersection of the dashed line and the *x*-axis up to the curve is $f(x)$. Again, this is a simple idea that we need to remember as it will become increasingly important as we move through the course.

Which curves do we need to be able to sketch quickly and without a calculator? The list is not long but we need to be able to sketch: $f(x)=x^2$, $\sqrt{x}$, $f(x)=x^3$, $f(x)=\ln x$, $f(x)=e^x$, $f(x)=\frac{1}{x}$, any linear function and any quadratic.

In addition to the list above, we need to be able to handle vertical and horizontal translations. We will address translations in detail in the next section. The basic idea is that if we know what $f(x)=\frac{1}{x}$ looks like, we know what $g(x)=\frac{1}{x-3}+4$ looks like. The graph of g is the graph of f shifted 3 units to the right and 4 units up.

A few words about quadratics before we move on. Any quadratic equation can be manipulated to look like $y-k=a(x-h)^2$ or $a(y-k)^2=x+h$. If we need to sketch the graph of a quadratic, we first rewrite it in one of these two forms. This process usually requires that we complete the square. Completing the square is one of our essential algebra skills that we will use over and over again all year long.

Piecewise functions

You are likely to see more piecewise functions this year than in any previous mathematics course. For now, you need to be able to sketch simple piecewise functions and find function values. One of the most common piecewise functions involves absolute value. <u>All functions that involve absolute value are piecewise functions</u> and we need to be able to write them as such. Consider $f(x)=|x+5|$. To write this as a piecewise function, we first need to know the "breaking point"—the point where the graph of the function has a cusp. To determine this point, set the expression inside the absolute value bars equal to zero and solve. For $f(x)=|x+5|$, we set $x+5=0$ and obtain $x=-5$. We can now rewrite f as follows:

$$f(x)=|x+5|=\begin{cases} x+5 & \text{if } x\geq -5 \\ -x-5 & \text{if } x<-5 \end{cases}$$

Now that the absolute value bars have been removed, the function is in a more useable form. Throughout the course, whenever you are faced with a function involving absolute value, you will first rewrite it as a piecewise function.

Odd and even functions

A quick review of a couple of definitions should be enough to bring you up to speed.

If $f(x)=f(-x)$, f is even. Even functions are symmetric with respect to the y-axis.
If $f(-x)=-f(x)$, f is odd. Odd functions are symmetric with respect to the origin.

Typically, one of the first times students run into difficulty with "presentation" is with problems involving odd or even functions. Please take a close look at how these problems are presented.

Example 4

Determine if the following function is odd, even or neither: $f(x) = x^5 + x$.

$$\text{Since } f(x) = x^5 + x,$$
$$f(-x) = (-x)^5 + (-x)$$
$$= -x^5 - x$$
and
$$-f(x) = -x^5 - x$$

Since $f(-x) = -f(x)$, f is odd.

Notice that we did not end the problem by simply stating, "f is odd". If we did, the reader of our problem is likely to ask, "Why?". You must state the definition as part of your answer!

Example 5

Determine if the following function is odd, even or neither: $g(x) = x^3 - 5x^2$.

$$\text{Since } g(x) = x^3 - 5x^2,$$
$$g(-x) = (-x)^3 - 5(-x)^2$$
$$= -x^3 - 5x^2$$
and
$$-g(x) = -x^3 + 5x^2$$

Since $g(x) \neq g(-x)$, g is not even.
Since $-g(x) \neq g(-x)$, g is not odd.
$\therefore g$ is neither odd nor even.

Note that we did not simply find $g(-x)$ and $-g(x)$ and then write "neither" as an answer. We started by rewriting the function we are going to work with, we found $g(-x)$ and $-g(x)$ and then we clearly and completely stated our conclusion. It may take some time, but you will all get very good at this with practice.

The value of knowing whether a function is odd or even is in the symmetry. If you are analyzing a function and can determine that it is even, you only need to analyze "half" of the function—the rest of the function behaves in a symmetric manner on the other side of the y-axis.

Operations on functions

Functions are mathematical objects which can be added, subtracted, multiplied and divided. In addition functions can be *composed*. The four basic operations are simple enough but let's look at a few examples of composition and finding the domain of a composition.

Example 6

Given $f(x)=\frac{1}{x}$ and $g(x)=\frac{1}{x-2}$, find $f(g(x))$ and its domain.

Finding $f(g(x))$

$$f(g(x)) = f\left(\frac{1}{x-2}\right) = \frac{1}{\frac{1}{x-2}} = x-2$$

$$\therefore\ f(g(x)) = x-2$$

Finding the domain of *f*(*g*(*x*))

Although $x-2 \ \exists\ \forall\ x$, the domain of *g* is $(-\infty,2)\cup(2,\infty)$ and so the domain of *f*(*g*(*x*)) is $(-\infty,2)\cup(2,\infty)$.

Important note about the domain of a composition and presentation:

The domain of a composition must be a subset of the domain of the "inside" function. Suppose we composed two functions and the expression resulting from the composition exists on $(-\infty,9)\cup(9,\infty)$ and the domain of the "inside" function is $(-\infty,3)\cup(3,\infty)$, the domain of the composition will be $(-\infty,3)\cup(3,9)\cup(9,\infty)$.

In Example 6, the composition resulted in the expression $x-2$, which taken by itself would exist for all *x*, but since the domain of *g* was $(-\infty,2)\cup(2,\infty)$, the domain of the composition must be $(-\infty,2)\cup(2,\infty)$. A common student error is to begin the domain discussion with " *f*(*g*(*x*)) exists for all *x*…". This almost always leads to a contradictory statement. If a student states, " $f(g(x))\ \exists\ \forall\ x$ but the domain of g is $(-\infty,2)\cup(2,\infty)$ so the domain of *f*(*g*(*x*)) is $(-\infty,2)\cup(2,\infty)$.", they have made a contradictory statement. You cannot say that a function exists for all *x* and then state that its domain is something different than all *x*. The way to avoid this error is to begin all discussions of the domain of a composition with the expression that is the result of the composition. Do not call it *f*(*g*(*x*)). Instead, first state the interval or intervals where the expression exists, then state the domain of the inside function, intersect the two and end with "therefore the domain of *f*(*g*(*x*)) is…".

Also note that this problem consists of two parts—finding the composition and then finding the composition's domain—and each part of the solution is titled appropriately. You <u>must</u> tell the reader what you are about to do. A reader should never have to read your mind to determine what you are about to do. Presentation counts. It will always count.

Example 7

Given $f(x)=x+2$ and $g(x)=\frac{1}{x}$, find $f(g(x))$ and its domain.

<u>Finding $f(g(x))$</u>

$$\begin{aligned} f(g(x)) &= f\left(\frac{1}{x}\right) \\ &= \frac{1}{x}+2 \\ &= \frac{1+2x}{x} \end{aligned}$$

$$\therefore\ f(g(x))=\frac{1+2x}{x}$$

<u>Finding the domain of *f*(*g*(*x*))</u>

$\frac{1+2x}{x}$ does not exist when $x=0$ and the domain of g is $(-\infty,0)\cup(0,\infty)$, thus the domain of $f(g(x))$ is $(-\infty,0)\cup(0,\infty)$.

Types of Functions, Translations and Scaling

Introduction

In this section we will first review a few families of functions. Throughout the course you will hear statements like "If *f* is a rational function…", or, "For polynomial functions…", and it will be important that you immediately know the form and basic behavior of the function that is being talked about. There are theorems which will only apply to specific types of functions so when you hear "rational function", you need to know exactly what a rational function is.

We will also address translations and scaling. The graph of a function is translated if it is moved horizontally or vertically. Scaling involves a function being compressed or stretched horizontally or vertically. It will be more important for you to have a good grasp of translations rather then scalings. In this course, we are more likely to encounter functions that have been translated rather than stretched or compressed. Knowledge of translations and scalings will allow you to sketch graphs of a wide variety of functions. For example, if you know what the graph of $y=\sqrt{x}$ looks like, and you understand translations, you will immediately know what the graph of $y=\sqrt{x-7}$ or $y=\sqrt{x}+3$ looks like. Complete facility with translations is essential.

Types of functions

Constant functions

- Constant functions are functions of the form $f(x)=c$ where *c* is a constant.
- The domain of any constant function is all the set of all real numbers.
- The graph of any constant function is a horizontal line.
- Examples: $f(x)=8,\ f(x)=e,\ g(x)=-1$

Power functions

- Power functions are functions of the form $f(x)=x^a$ where *a* is a constant.
- If $a=-1$, the graph of the function will be a hyperbola.
- If $a=\frac{1}{n}$ where n is a positive integer, the function is a root function.

Polynomial functions

- Polynomial functions are functions of the form $f(x)=a_n x^n + a_{n-1}x^{n-1}+...+a_2x^2+a_1x+a_0$.
- The domain of a polynomial function is the set of all real numbers.
- For $n=1$, f is a linear function
- For $n=2$, f is a quadratic function
- For $n=3$, f is a cubic function
- For $n=4$, f is a quartic function
- For $n=5$, f is quintic function (we do not name functions above degree five)

Rational functions

- Rational functions are functions of the form $f(x) = \dfrac{P(x)}{Q(x)}$ where both P and Q are polynomials. This is a very important definition. For some reason, students think that any function, which has variables in a denominator, is a rational function. This is not true. The function $f(x) = \dfrac{x-4}{\sqrt{x+8}}$ is NOT a rational function because its denominator is not a polynomial!
- The domain of a rational function is the set of all reals such that $Q(x) \neq 0$.

Algebraic functions

- Algebraic functions are functions that are constructed by performing algebraic operations (addition, subtraction, multiplication, division and taking roots) on polynomials. The function $f(x) = \dfrac{x-4}{\sqrt{x+8}}$ is an algebraic function. All rational functions are algebraic…but not all algebraic functions are rational.
- We will spend a great deal of time in this course learning how to analyze algebraic functions.

Trigonometric functions

- We'll do a complete review of these in a later section. Let it suffice to say that we LOVE the trigonometric functions! Why? Because they're periodic. It's their periodicity that makes them predictable and easy to work with.

Exponential functions

- Exponential functions are functions of the form $f(x) = a^x$ where a is a positive constant. Do not confuse exponential functions with power functions! We will study exponential functions in great detail later in the course.

Logarithmic functions

- Logarithmic functions are functions of the form $f(x) = \log_a x$ where a is a positive constant. Logarithmic functions are the inverse of the exponential functions and will be studied in detail when we address exponential functions.

Vertical and horizontal translations

You've spent time already in your precalculus courses studying vertical and horizontal translations. In the table below we have listed the various translations, which can be performed on a function. Make sure you know and understand these translations—they will make your work easier. We will be sketching thousands of functions and the better you understand translations, the easier the course will be.

Suppose $c > 0$.

The graph of $y = f(x) + c$ is the graph of $y = f(x)$ shifted *c* units upward.
The graph of $y = f(x) - c$ is the graph of $y = f(x)$ shifted *c* units downward.
The graph of $y = f(x + c)$ is the graph of $y = f(x)$ shifted *c* units to the left.
The graph of $y = f(x - c)$ is the graph of $y = f(x)$ shifted c units to the right.

Here are some examples of translations:

- The graph of $f(x) = \frac{1}{x-6}$ will be the graph of $f(x) = \frac{1}{x}$ translated 6 units to the right.
- The graph of $g(x) = x^3 - 4$ will be the graph of $g(x) = x^3$ translated 4 units down.
- The graph of $h(x) = \ln(x+2)$ will be the graph of $h(x) = \ln x$ translated 2 units to the left.

Scaling (stretching and compressing)

Scaling is sometimes more difficult to talk about than translations. This is due to the confusion caused by terms like "compressed vertically" vs. "stretched horizontally" and "stretched vertically" and "compressed horizontally". Visually, these pairs of expressions seem to mean the same thing…but technically they are different. If you can get to the point where you can tell the difference between $f(x) = 5x^2$, $f(x) = \frac{1}{5}x^2$ and $f(x) = x^2$ you'll be fine. The details are in the table below.

Suppose $c > 1$.

The graph of $y = c\,f(x)$ will be the graph of $y = f(x)$ stretched vertically by a factor of *c*.

The graph of $y = \frac{1}{c} f(x)$ will be the graph of $y = f(x)$ compressed vertically by a factor of *c*.

The graph of $y = f(cx)$ will be the graph of $y = f(x)$ compressed horizontally by a factor of *c*.

The graph of $y = f\left(\frac{1}{c}x\right)$ will be the graph of $y = f(x)$ stretched horizontally by a factor of *c*.

Reflections

In addition to the translations and scalings discussed above, graphs of functions can be reflected about axes.

The graph of $y = -f(x)$ will be the graph of $y = f(x)$ reflected about the *x*-axis.
The graph of $y = f(-x)$ will be the graph of $y = f(x)$ reflected about the *y*-axis.
The graph of $y = |f(x)|$ will be the graph of $y = f(x)$ in which all the portions below the *x*-axis are reflected across the *x*-axis.

Trigonometry Review

Introduction

The six trigonometric functions are examples of transcendental functions. If you are like most students, as soon as you hear "trig" your mouth dries up, your brain shuts down and you feel like you're about to be sent back to an area you never really understood in the first place. Relax. The trigonometric functions are actually very well-behaved functions. The phrase "well-behaved" is an actual phrase used by mathematicians to describe functions that are predictable, smooth functions. The trigonometric functions are among the most well-behaved functions. This is because they are all periodic. If you know how they behave over one period, you know their behavior everywhere! The amount of trigonometry you need to know to be successful in this course is minimal…but you need to know it! You need to be able to recall it instantly—otherwise, problems that are actually trivial will become "undoable". Basically, you need to be able to use reference angles, solve simple equations involving the trigonometric functions and you need to know (memorize) a dozen or so identities.

Radians, radians, radians

Every problem you encounter this year, which involves trigonometry, will be done in radians. All the trigonometric functions are defined in terms of radians. Let's start by finally understanding what a radian is and why we use them.

First of all, radians are real numbers, degrees are not. Degrees are a relic from the Babylonians who had a base 60 number system and decided that it would be nice to divide a circle into 360 equal parts—mostly because 360 is evenly divisible by so many different integers. A degree is an arbitrary unit, not based on any measurement. A radian however, is a real number. A radian is the measure of the length of an arc. Rather than measuring an angle by somehow measuring how spread apart the two sides of the angle are, radians measure the length of the arc the angle subtends. This is why so many familiar radian measures involve π. The circumference of a circle is calculated by $C = 2\pi r$. We normally refer to the "unit circle" which is a circle of radius one—one foot, one inch, one centimeter…it doesn't matter. If the radius is 1, the circumference is 2π. A right angle would take us $\frac{1}{4}$ of the way around our unit circle—or $\frac{\pi}{2}$ radians. If the circle were one foot in radius, we would have to travel $\frac{\pi}{2}$ feet to get one-fourth of the way around. Similarly, going half way around the circle means we would travel π feet around the circle. Since it doesn't matter what units are used to measure the radius, we use radians to refer to the length of the arc.

Identities you need to know

The following identities need to be ***memorized***:

$$\tan x = \frac{\sin x}{\cos x}$$
$$\cot x = \frac{\cos x}{\sin x}$$
$$\sec x = \frac{1}{\cos x}$$
$$\csc x = \frac{1}{\sin x}$$
$$\sin(-x) = -\sin x \ (\text{sine is an odd function})$$
$$\cos(-x) = \cos x \ (\text{cosine is an even function})$$
$$\sin^2 x + \cos^2 x = 1$$
$$1 + \cot^2 x = \csc^2 x$$
$$\tan^2 x + 1 = \sec^2 x$$
$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$
$$\sin(a-b) = \sin a \cos b - \cos a \sin b$$
$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$
$$\cos(a-b) = \cos a \cos b + \sin a \sin b$$
$$\sin 2x = 2\sin x \cos x$$
$$\cos 2x = \cos^2 x - \sin^2 x$$
$$\cos 2x = 2\cos^2 x - 1$$
$$\cos 2x = 1 - \sin^2 x$$

Conversion between radians and degrees

It is often useful, especially when using reference angles, to do a problem in degrees instead of radians. To convert, use the following:

$$\text{Degrees into radians: degrees} \bullet \frac{p}{180}$$
$$\text{Radians to degrees: radians} \bullet \frac{180}{p}$$

Standard values

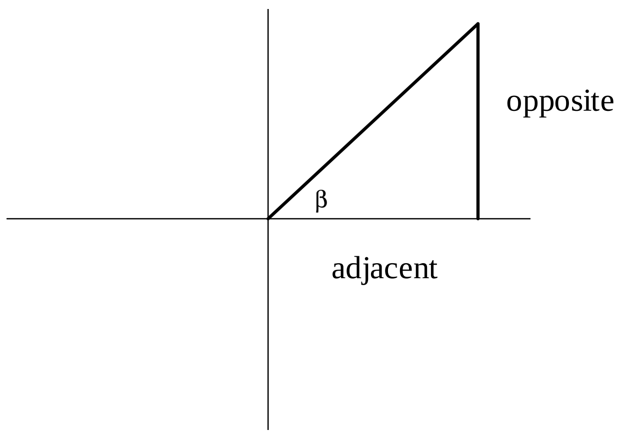
We will need to have memorized (or be able to calculate) the value of the six trigonometric functions for any angle that is a multiple of 30 or 45 degrees—in the positive or negative direction. These include:

$$0, \frac{p}{6}, \frac{p}{4}, \frac{p}{3}, \frac{p}{2}, \frac{2p}{3}, \frac{3p}{4}, \frac{5p}{6}, p, \frac{7p}{6}, \frac{5p}{4}, \frac{4p}{3}, \frac{3p}{2}, \frac{5p}{3}, \frac{7p}{4}, \frac{11p}{6}, 2p.$$

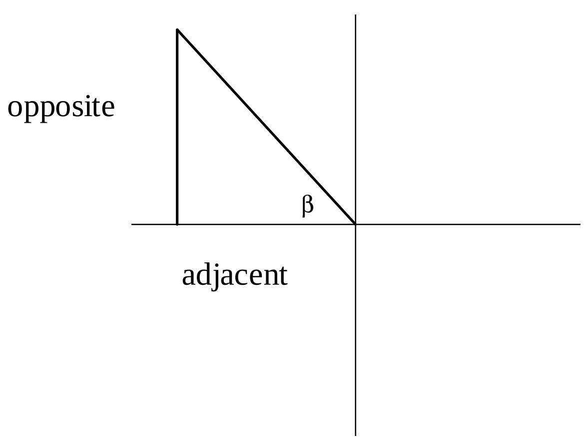
They do not need to be memorized. These values are easy to calculate using reference angles, special triangles and the unit circle. For multiples of 30 and 45, use reference angles and either a 30-60-90 or 45-45-90 triangle. For multiples of 90, use the unit circle.

Using reference angles

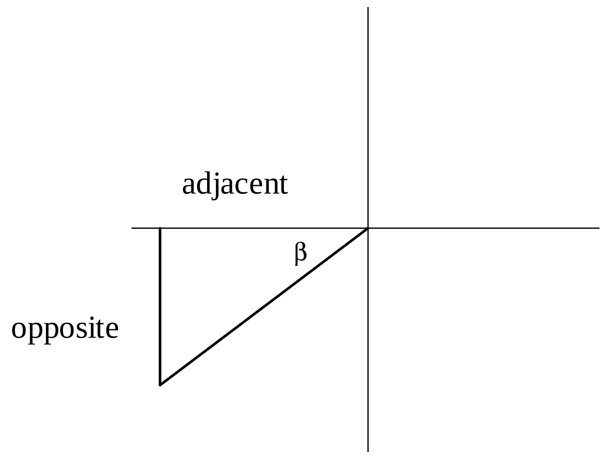
To use reference angles, first convert your angle measurement to degrees. Remember, we only need to use reference angles if you have a multiple of 30 or 45. If the angle is in the first quadrant, our reference angle and triangle will look like this:



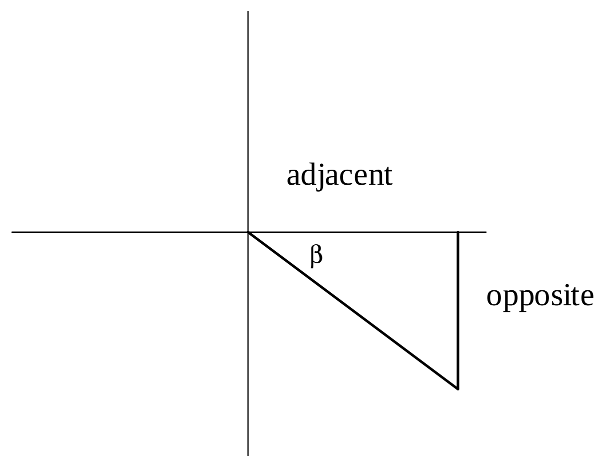
If our angle is in the second quadrant, our reference angle and triangle will look like this:



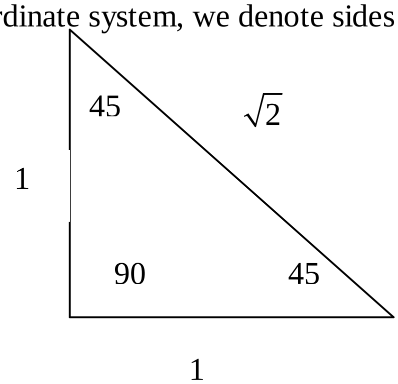
If our angle is in the third quadrant, our reference angle and triangle will look like:

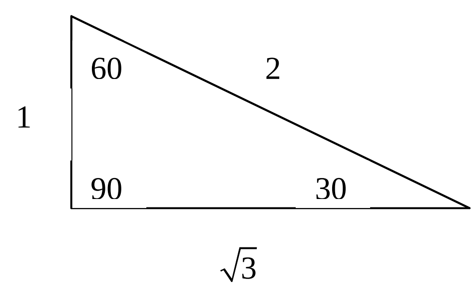


Again, in the fourth quadrant, our reference angle and triangle will look like:



After drawing our reference angle and triangle, we label the sides of the triangle using one of the two special triangles illustrated below. Depending on which quadrant our triangle is in, one or both of the sides will be "negative". A distance cannot actually be negative, but when the triangle is drawn on the coordinate system, we denote sides which are drawn in a negative direction as negative.





Now that we have our angle converted and our triangle drawn and labeled, we make use of the following relationships:

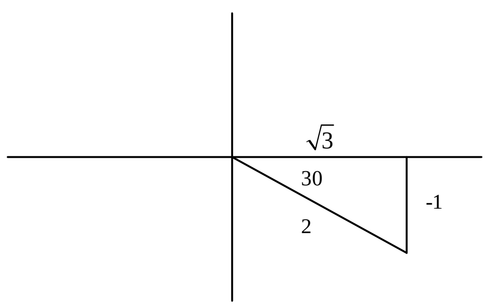
$$\sin b = \frac{opposite}{hypotenuse}$$
$$\cos b = \frac{adjacent}{hypotenuse}$$
$$\tan b = \frac{opposite}{adjacent}$$
$$\cot b = \frac{adjacent}{opposite}$$
$$\sec b = \frac{hypotenuse}{adjacent}$$
$$\csc b = \frac{hypotenuse}{opposite}$$

Example 1

Find $\sin\left(\frac{11p}{6}\right)$

We begin by converting our angle to degrees: $\frac{11p}{6} = 330^{0}$

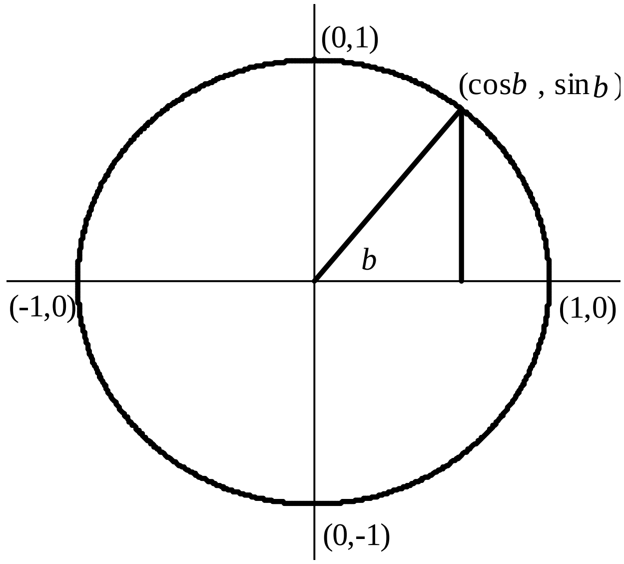
Next we draw our reference angle and triangle.



Since sine is the opposite side divided by the hypotenuse, $\sin\left(\frac{11p}{6}\right) = -\frac{1}{2}$

Using the unit circle

If we need to find the value of a trigonometric function for an angle that is a multiple of 90 we will use the unit circle. The unit circle (a circle of radius one) is used to define the trigonometric functions. Any point on the unit circle can be labeled $(\cos b, \sin b)$ as seen in the diagram below.



Example 2

Find $\sin p$.

p radians is 180 degrees.
Sine is given by the second member of the coordinate pair, so $\sin p = 0$.

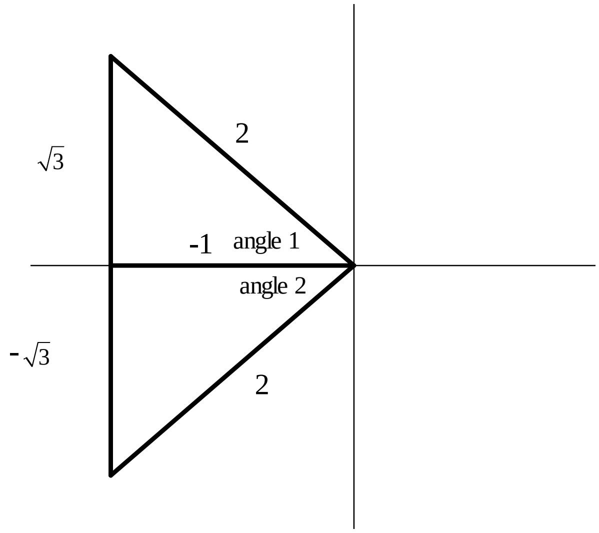
More often than not you will be asked to find an angle whose trigonometric value is given. This happens all the time when we solve equations involving trigonometric functions. Instead of being given an angle and finding out what its sine is, for example, you will be given the sine of an angle and then be asked to find the angle.

Example 3

Find the angle(s) in the interval $[0, 2p]$ whose cosine is $-\frac{1}{2}$.

We need to draw a reference angle and triangle so that the cosine of the reference angle is $-\frac{1}{2}$.
We know that the cosine of an angle is the adjacent side divided by the hypotenuse.
We leave the negative sign with the numerator (the adjacent side) since, by convention, the hypotenuse is always positive.
Adjacent sides are drawn along the x-axis, opposite sides are drawn perpendicular to the x-axis.

Here is our diagram. Notice that there are two reference angles that have a cosine equal to $-\frac{1}{2}$.



The third side of these two triangles is $\sqrt{3}$ and the triangle is a 30-60-90 with angle 1 and angle 2 both equal to 60 degrees.

Now, we can say that the two angles in $[0,2p]$ whose cosine is $-\frac{1}{2}$ are $\frac{2p}{3}$ and $\frac{4p}{3}$.

Note that if the original interval was $[-2p,0]$, the answers would be $-\frac{4p}{3}$ and $-\frac{2p}{3}$. We get these angles by moving in a negative direction (clockwise).

Solving trigonometric equations

We will rarely deal with any but the most simple of trigonometric equations.

Example 4

Solve $\sin 2x = 0$ on $[0,2p]$.

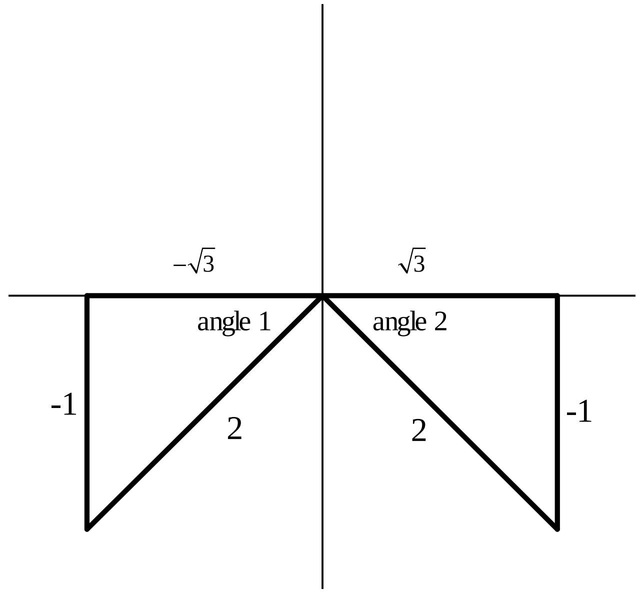
First, we ask ourselves, "When is the sine of ANY angle equal to zero?".
Using the unit circle we see that sine will be zero when the angle is 0 or p so
$\sin 2x = 0$ when $2x = 0$ or $2x = p$ or $2x = 2p$ or $2x = 3p$ or $2x = 4p$

$$\therefore x = 0 \text{ or } x = \frac{p}{2} \text{ or } x = p \text{ or } x = \frac{3p}{2} \text{ or } x = 2p$$

Example 5

Solve $\sin 2x = -\frac{1}{2}$ on $[0,2p]$.

First we determine when the sine of any angle is equal to $-\frac{1}{2}$. We do this by drawing a diagram.



The third side of these two triangles is $\sqrt{3}$ and the triangle is a 30-60-90 with angle 1 and angle 2 both equal to 30 degrees.

These two reference angle correspond to $\frac{7p}{6}$ and $\frac{11p}{6}$.

We can now say,

$$\sin 2x = -\tfrac{1}{2} \text{ when } 2x = \frac{7p}{6} \text{ or } 2x = \frac{11p}{6} \text{ or } 2x = \frac{19p}{6} \text{ or } 2x = \frac{23p}{6}$$

$$\therefore x = \frac{7p}{12} \text{ or } x = \frac{11p}{12} \text{ or } x = \frac{19p}{12} \text{ or } x = \frac{23p}{12}$$

Inequalities and Absolute Value

Introduction

In this section we will discuss techniques used to solve a wide variety of problems involving inequalities, absolute value or both. We will encounter problems of this nature throughout the course so it is essential that we can handle them with ease.

Whenever the answer to a problem is an interval, like $x \geq -9$ or $2 < x < 7$, we will use interval notation. The table below should refresh your memory.

Inequality notation	Interval notation
$a < x < b$	(a, b)
$a \leq x < b$	$[a, b)$
$a < x \leq b$	$(a, b]$
$a \leq x \leq b$	$[a, b]$
$x > a$	(a, ∞)
$x \geq a$	$[a, \infty)$
$x < a$ or $x > b$	$(-\infty, a) \cup (b, \infty)$
$x < a$ or $x \geq b$	$(-\infty, a) \cup [b, \infty)$
$x \leq a$ or $x > b$	$(-\infty, a] \cup (b, \infty)$
$x \leq a$ or $x \geq b$	$(-\infty, a] \cup [b, \infty)$
All the reals	$(-\infty, \infty)$

Notice that an open parentheses is always used with the infinity symbol.

A special note about solving equations involving rational expressions

You've solved equations involving rational expressions since your first algebra course. The technique you've been taught is to multiply both sides of the equation by a common denominator—thus "eliminating" the denominator. Consider the equation $\frac{5x}{x+6} = 2$. Using this technique, you would multiply both sides by $x+6$ which would yield

$$\begin{aligned} 5x &= 2(x+6) \\ 5x &= 2x+12 \\ 3x &= 12 \\ x &= 4 \end{aligned}$$

When you multiply both sides of an equation by a variable expression you must be sure to check your solutions to make sure they satisfy the original statement. Multiplying both sides of an equation by a variable expression may produce extraneous solutions. Consider $x+3=\dfrac{-2x^2+7x-3}{x-3}$. If both sides are multiplied by $x-3$, we obtain

$$3x^2-7x-6=0$$
$$(3x+2)(x-3)=0$$
$$x=-\frac{2}{3} \text{ or } x=3$$

Checking solutions reveals that $x=3$ does not solve the original problem and thus is an extraneous solution.

In this course, many of our problems will be loaded with subtleties and nuances. We will have enough to think about without having to worry about whether we've introduced extraneous solutions as part of solving the larger problem.

To avoid the possible introduction of extraneous solutions, in this course we will use a different technique to solve equations involving rational expressions.

The technique, which may be new to you, does not involve algebra you do not already know. So, in this course, to solve ANY equation involving rational expressions you will

- bring all terms to one side
- get a common denominator
- set the numerator equal to zero to find solutions

This technique eliminates any possibility of introducing extraneous solutions. It also allows us to determine easily where a particular expression fails to exist. Thus, to solve $\dfrac{x-3}{x+4}=5$, we will first bring the 5 to the right side and find a common denominator.

$$\frac{x-3}{x+4}=5$$
$$\frac{x-3}{x+4}-5=0$$
$$\frac{x-3-5(x+4)}{x+4}=0$$
$$\frac{-4x-23}{x+4}=0$$

$$\text{Now, } \frac{-4x-23}{x+4}=0 \text{ when } -4x-23=0$$
$$x=-\frac{23}{4}$$

This may seem like a longer procedure, and for some problems, it is—but it eliminates any need to check for extraneous solutions. It is the technique you will be required to use.

Inequalities without absolute value

We will begin with the most elementary of problems and work our way to more challenging problems.

Example 1

Solve: $2+3x<5x+8$

Of course you can do this one! The only thing you may want to do, in the process of solving, is isolate the variable on the left side. This makes the conversion to interval notation easier.

$$2+3x<5x+8$$
$$-2x<6$$
$$x>-3$$
$$\therefore x\in(-3,\infty)$$

Example 2

Solve: $\dfrac{7}{x}>2$

We may be tempted to multiply both sides by *x* here but we need to be careful. As the problem stands, $x=0$ cannot be a solution because $\dfrac{7}{x}$ is undefined at $x=0$. If we multiply both sides by *x*, we will get an inequality for which $x=0$ is a solution. The lesson here is…***<u>never multiply or divide both sides of an inequality by a variable expression</u>*** because we do not know whether the expression is positive or negative and so we do not know if we should change the direction of the inequality! Multiplying both sides of an inequality by a variable expression is allowed if we want to set up the necessary cases—but that unnecessarily complicates the process of solving.

Instead we will use the technique introduced previously, bring all terms to the left side, get a common denominator and proceed.

$$\frac{7}{x}>2$$
$$\frac{7}{x}-2>0$$
$$\frac{7-2x}{x}>0$$

Now we will make a chart to determine on which intervals the expression $\dfrac{7-2x}{x}$ is greater than zero.

First we determine TWO ITEMS: (1) where the expression fails to exist and (2) where the expression is equal to zero. Both of these must be considered. We will perform this little two-step procedure thousands of times throughout the course…so get used to it! The expression will be equal to zero when the numerator is zero and will fail to exist if the denominator is zero…it's that easy.

(1) $\frac{7-2x}{x} \nexists$ when $x = 0$

(2) $\frac{7-2x}{x} = 0$ when $7 - 2x = 0$

$$x = \frac{7}{2}$$

We will now make a chart using $x = 0$ and $x = \frac{7}{2}$.

	$\frac{7-2x}{x}$
$(-\infty, 0)$	$-$
$x = 0$	$\nexists$
$\left(0, \frac{7}{2}\right)$	$+$
$x = \frac{7}{2}$	0
$\left(\frac{7}{2}, \infty\right)$	$-$

$$\therefore x \in \left(0, \frac{7}{2}\right)$$

Note that if we did multiply by x in the first step we would have

$$7 > 2x$$

$$\frac{7}{2} > x$$

$$x < \frac{7}{2}$$

This solution, $x < \frac{7}{2}$, is incorrect because it includes $x = 0$ and all the negative numbers as solutions…but the original problem does not exist at $x = 0$ and is not satisfied by any negative number.

Example 3

Solve: $\frac{x}{x-3} < 4$

Remember the lessons learned with the last problem! Bring everything to one side, get a common denominator and use a chart.

$$\frac{x}{x-3} < 4$$
$$\frac{x}{x-3} - 4 < 0$$
$$\frac{x-4(x-3)}{x-3} < 0$$
$$\frac{-3x+12}{x-3} < 0$$

(1) $\frac{-3x+12}{x-3} \nexists$ when $x = 3$

(2) $\frac{-3x+12}{x-3} = 0$ when $-3x+12 = 0$

$x = 4$

	$\frac{-3x+12}{x-3}$
$(-\infty, 3)$	$-$
$x = 3$	$\nexists$
$(3, 4)$	$+$
$x = 4$	0
$(4, \infty)$	$-$

$$\therefore x \in (-\infty, 3) \cup (4, \infty)$$

Example 4

Solve: $4 < 3x - 2 < 10$

We've been taught two ways to approach a combined inequality. One technique is to operate on all three parts at the same time. The other approach is to separate the statement into two parts.

$$4 < 3x - 2 < 10$$
$$6 < 3x < 12$$
$$2 < x < 4$$
$$\therefore x \in (2, 4)$$

or

$$4 < 3x - 2 < 10$$
$$4 < 3x - 2 \text{ and } 3x - 2 < 10$$
$$6 < 3x \text{ and } 3x < 12$$
$$2 < x \text{ and } x < 4$$
$$\therefore x \in (2, 4)$$

The only time when the first technique will not work is if we have variables in more than one of the three expressions. If this happens, we cannot isolate the variable in the "center"…we end up chasing it around the inequality. In this case, the second technique must be used.

Example 5

Solve: $x^3 + 1 > x^2 + x$

This is an example of what we will call a "higher order" inequality. To solve we will bring all the terms to the left side. We will then make a chart to determine the solution. Remember, every chart begins with TWO statements…one tells us where the expression is equal to zero and the other tells us where the expression fails to exist (if anywhere).

$$x^3 + 1 > x^2 + x$$
$$x^3 - x^2 - x + 1 > 0$$
$$(x+1)(x-1)(x-1) > 0$$

(1) $x^3 - x^2 - x + 1 \ \exists \forall x$

(2) $x^3 - x^2 - x + 1 = 0$ when $x = -1$ or $x = 1$

	$x^3 - x^2 - x + 1$
$(-\infty, -1)$	$-$
$x = -1$	0
$(-1, 1)$	$+$
$x = 1$	0
$(1, \infty)$	$+$

$$\therefore x \in (-1,1) \cup (1, \infty)$$

It is essential that we always use precise, accurate, correct mathematics. A common student error is to write

$$x^3 + 1 > x^2 + x$$
$$x^3 - x^2 - x + 1 > 0$$
$$(x+1)(x-1)(x-1) > 0$$
$$x = -1 \text{ or } x = 1$$

This is incorrect mathematics! The solutions stated in the final step are NOT solutions to the inequality. If $x = -1$ or $x = 1$, the expression is zero, not greater than zero!

Absolute value

Before we address inequalities involving absolute value, we should quickly review absolute value itself. First, a simple definition:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

When we think of this definition, we want to avoid thinking that it only holds for "*x*". The *x* could be anything. In our minds we want to see a definition which looks more like:

$$|anything| = \begin{cases} anything & \text{if } anything \geq 0 \\ -anything & \text{if } anything < 0 \end{cases}$$

Here are several important absolute value theorems which we use to solve inequalities involving absolute value:

$$|x| < a \Leftrightarrow x < a \text{ and } x > -a$$
$$|x| \leq a \Leftrightarrow x \leq a \text{ and } x \geq -a$$

$$|x| > a \Leftrightarrow x > a \text{ or } x < -a$$
$$|x| \geq a \Leftrightarrow x \geq a \text{ or } x \leq -a$$

Absolute value is often defined in terms of distance. $|x|$ is the distance from the origin to *x*. Using this definition and the theorems above, let's consider the inequality $|x-3| < 5$.

$$|x-3| < 5$$
$$x-3 < 5 \text{ and } x-3 > -5$$
$$x < 8 \text{ and } x > -2$$

Notice that the solution includes all the numbers between –2 and 8. The number halfway between these two numbers is 3. The distance from –2 to 3 is 5, the distance from 3 to 8 is 5 so the inequality $|x-3| < 5$ describes all the *x*'s that are within 5 units of 3.

Similarly, the inequality $|x-7| < 9$ describes all the *x*'s that are within 9 units of $7 \ldots$or $(-2,16)$.
The inequality $|x+2| < 13$ describes all the *x*'s that are within 13 units of –2 $\ldots$or $(-15,11)$.

We bring up this interesting way to "translate" these simple inequalities because soon we will be defining the term "limit" and if we can *read* the definition instead of just reciting the symbols, the definition will make much more sense.

Solving inequalities involving absolute value

Again, we will consider several different types of inequalities involving absolute value—starting with the simpler and moving to the more complex.

Example 6

Solve: $|3x+2|>5$

$$|3x+2|>5$$
$$3x+2>5 \text{ or } 3x+2<-5$$
$$3x>3 \text{ or } 3x<-7$$
$$x>1 \text{ or } x<-\tfrac{7}{3}$$
$$\therefore x\in\left(-\infty,-\tfrac{7}{3}\right)\cup(1,\infty)$$

Example 7

Solve: $|x+4|=|2x-6|$

$|x+4|=|2x-6|$ is equivalent to $|x+4|=2x-6$
(A demonstration of this can be found at the end of the section.)

$$|x+4|=2x-6$$
$$x+4=2x-6 \text{ or } x+4=-(2x-6)$$
$$x=10 \text{ or } x=\tfrac{2}{3}$$

Our last example will involve an inequality with absolute value and a nonlinear expression.

Example 8

Solve: $\left|\dfrac{6-5x}{x+3}\right| \le \dfrac{1}{2}$

$$\left|\frac{6-5x}{x+3}\right| \le \frac{1}{2}$$

$$\frac{6-5x}{x+3} \le \frac{1}{2} \text{ and } \frac{6-5x}{x+3} \ge -\frac{1}{2}$$

$$\frac{6-5x}{x+3} - \frac{1}{2} \le 0 \text{ and } \frac{6-5x}{x+3} + \frac{1}{2} \ge 0$$

$$\frac{9-11x}{2x+6} \le 0 \text{ and } \frac{15-9x}{2x+6} \ge 0$$

(1) $\dfrac{9-11x}{2x+6} \nexists\ x=-3$ and (1) $\dfrac{15-9x}{2x+6} \nexists$ when $x=-3$

(2) $\dfrac{9-11x}{2x+6} = 0$ when $x = \dfrac{9}{11}$ and (2) $\dfrac{15-9x}{2x+6} = 0$ when $x = \frac{5}{3}$

	$\frac{9-11x}{2x+6}$
$(-\infty,-3)$	$-$
$x=-3$	$\nexists$
$\left(-3,\frac{9}{11}\right)$	$+$
$x=\frac{9}{11}$	0
$\left(\frac{9}{11},\infty\right)$	$-$

and

	$\frac{15-9x}{2x+6}$
$(-\infty,-3)$	$-$
$x=-3$	$\nexists$
$\left(-3,\frac{5}{3}\right)$	$+$
$x=\frac{5}{3}$	0
$\left(\frac{5}{3},\infty\right)$	$-$

$\dfrac{9-11x}{2x+6} \le 0$ when $x \in (-\infty,-3) \cup \left[\frac{9}{11},\infty\right)$ and $\dfrac{15-9x}{2x+3} \ge 0$ when $x \in \left(-3,\frac{5}{3}\right]$

$$\therefore x \in \left[\frac{9}{11},\frac{5}{3}\right]$$

Summary

When solving inequalities:

- If the problem involves only linear expressions, we solve it like we solve linear equations
  - § variables to one side, constants to the other
  - § if you multiply or divide by a negative, change the direction of the inequality
- If the problem involves a non-linear, rational expression
  - § get everything on one side so we are comparing the expression to zero

  - § get a common denominator
  - § determine where the expression fails to exist
  - § determine where the expression is equal to zero
  - § make a chart
- If the problem involves a higher order expression
  - § get everything on one side so we are comparing the expression to zero
  - § determine where the expression fails to exist
  - § determine where the expression is equal to zero by factoring
  - § make a chart
- If the problem involves absolute value
  - § set up the appropriate cases using the definitions
  - § solve the resulting equations or inequalities

Demonstration that $|a| = |b|$ is equivalent to $|a| = b$

$$|a| = |b|$$

Now, if we let $\Delta = |b|$ we obtain

$$|a| = \Delta,$$

which by definition means

$$a = \Delta \text{ or } a = -\Delta$$

so

$$\Delta = a \text{ or } \Delta = -a$$

Now replace $\Delta = |b|$.

$$|b| = a \text{ or } |b| = -a$$

Again using definitions we can say

$$b = a \text{ or } b = -a \text{ or } b = -a \text{ or } b = a$$

Clearly, these statements can be restated as

$$a = b \text{ or } a = -b$$

This last statement is equivalent to $|a| = b$ which means we can always replace $|a| = |b|$ with $|a| = b$.
